THE

PHYSICAL SOCIETY

OF

LONDON.

PROCEEDINGS.

VOLUME XXVII.—PART V.

AUGUST 15, 1915.

Price to Non-Fellows, **4s.** *net, post free* **4/3**.

Annual Subscription, **20/-** *post free, payable in advance*

Published Bi-Monthly from December to August.

LONDON:
"THE ELECTRICIAN" PRINTING & PUBLISHING CO., LTD.,
1, 2 AND 3, SALISBURY COURT, FLEET STREET.

1915.

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XXXIII. *The Cohesion of Solids.* *By* Prof. Herbert Chatley, *D.Sc.* (*Lond.*), *M.I.C.E.I.*

Received December 13, 1914.

It is a remarkable fact that the force which links the particles of the earth to those of the distant sun is much better understood than that which connects those earth particles together. The most plausible hypothesis would seem to be that which identifies the two. Lord Kelvin favoured this view, but there are several difficulties.

Two salient facts arise from any investigation, viz. :—

(*a*) The attractive force which we term cohesion has a very minute range, but appears to be considerably greater than gravitation at minute distances.

(*b*) There is a repulsion between small particles of matter as well as an attraction.

In order to appreciate the second fact correctly it is necessary to make a preliminary conjecture as to the nature of the ultimate particles of matter. Elaborate investigation has shown that solid matter has a whole series of discontinuities—

1. Crystalline structure bounded by interfaces, with occasional minute cavities between the crystals. This structure is usually visible in microscopic sections. In the case of "amorphous" solids it may, perhaps, be assumed that the crystals are of molecular dimensions.

2. Molecules consisting of a geometrically arranged group of two or more atoms bound by electrostatic (?) forces (chemical affinity) immensely superior in intensity to gravitation or cohesion.

3. Atoms consisting of electrons, each atom forming a dynamic complex of immense stability.

4. Possible etheric sub-structure, such as Osborne Reynolds' "granules" of "negative matter."

It would appear that under ordinary circumstances the atoms behave as indefinitely rigid minute solids, so that discontinuity of substance or delay in the transmission of momentum arises solely from the fact that the molecules are separated by appreciable distances.

The elastic compressibility of solids shows that—

(*a*) The molecules or the atoms in the molecules may be brought closer together.

(*b*) Such closer proximity develops a mutual repulsion in the molecules, and "strain energy" is stored in the substance,

which will return the parts to their original positions (or nearly so) when the compressing force is removed.

Elastic tension similarly indicates the presence of a mutual attraction and separability.

A solid under no stress has its particles in a state of equilibrium. Disturbance of that state in the direction of separation or approximation is accompanied by a restoring force. Temporary application of stress is, therefore, followed after its removal by harmonic oscillations, which are at first congruous and then become degraded into heat, which is radiated or conducted away.

The application or removal of heat causes a difference in the equilibrium position of the particles. The addition of heat causes expansion, implying an increase in the mutual repulsion, diminution in the attraction, or both. Conversely, the subtraction of heat implies a decrease in the repulsion, increase in the attraction, or both. This phenomenon, together with certain others, has led physicists to believe that heat in solids consists in a vibration of the particles, the kinetic energy varying with the total heat. If the absolute zero of temperature is also the zero of heat, the mean kinetic energy of the particles varies approximately as the absolute temperature.

If it is assumed that the vibrations are linear, simple, harmonic oscillations (an assumption which is certainly not quite true),then we may write approximately :—

Total heat=number of molecules × mean kinetic energy per molecule.

$$\mathrm{JM}s\theta = n^3 \frac{m \cdot \pi^2 \cdot \Delta^2}{\mathrm{T}^2},$$

where M=mass of unit volume=$n^3 m$,

s=mean specific heat from zero absolute to θ,

θ=temperature (absolute), centigrade,

J=mechanical equivalent of heat, ergs per gram-calorie,

n^3=molecules in unit volume,

m=mass of a molecule,

Δ=mean amplitude of oscillation,

T=mean periodic time of oscillation.

$$\frac{\Delta}{\mathrm{T}} = \frac{1}{\pi}\sqrt{\mathrm{J}s\theta}.$$

The mean velocity of the particles is $\frac{4\Delta}{\mathrm{T}}$.

For example, steel at temperature of 300 deg. absolute (27°C., 80°F.),

$$\frac{\Delta}{T}=\frac{1}{\pi}\sqrt{4\cdot2\times10^{7}\times0\cdot11\times300}=11{,}850 \text{ cm. per second.}$$

$$v=\frac{4\Delta}{T}=47{,}400 \text{ cms./sec.}=1{,}555 \text{ ft./sec.}$$

A dull red heat corresponding to a temperature of about 600°C. (say 900 deg. abs.) will then involve a velocity of about 1·7 times the above, say 2,500 ft./sec., or 80,000 cms./sec. Optics shows that the vibrations in light of that colour have a frequency of about 400×10^{12} per second, so that on these assumptions

$$\Delta \text{ at 900 deg. abs.}=\frac{80{,}000}{400\times10^{12}\times4}=0\cdot5\times10^{-8}.$$

2Δ, the molecular swing, would then be $1\cdot0\times10^{-8}$ cm. which agrees moderately well with otherwise determined molecular dimensions.

Such oscillators would present great resistance to compression applied in the direction of oscillation, but, per se, their attraction would not be different from the usual gravitational one, and even at the small distance of $0\cdot5\times10^{-7}$ cm. the gravitational force is small.

The force of gravitation follows the law

$$t \text{ (dynes)}=\frac{Gm_1m_2}{d^2},$$

where $G=6\cdot6\times10^{-8}$; m_1, m_2 are the attracting masses (grammes); d is the distance, centre to centre, centimetres.

If we write $m_1=m_2=\frac{M}{n^3}$, where $n=\frac{1}{d}$,

$$t=G\frac{M^2}{n^4}=GM^2d^4,$$

where M=mass per unit volume=specific gravity.

Thus, in steel, specific gravity, say 8·0,

$$t=6\cdot6\times10^{-8}\times64\times(0\cdot5\times10^{-7})^4,$$

an excessively minute force, apparently quite incapable of explaining cohesion.

It seems difficult to understand how the molecules can be brought much closer together than the above-mentioned distance without developing repulsive forces greatly in excess of t.

At very low temperatures there is a contraction which may reduce the distance to perhaps half that given. Cohesion does increase at such low temperatures, but still the gravitational force seems insufficient.

Kelvin was of opinion that cohesion could be explained by supposing the actual mass of substance to be concentrated into volumes quite small as compared with the whole volume. The density at such centres would be extremely high, and this intense density would, he thought, so increase the force of gravity between pairs of molecules very close together as to account for cohesion without necessitating any deviation from the inverse square law ("Pop. Lect.," Vol. I., and "Proc." Roy. Soc. Edin., April 21, 1862, Vol. IV.).

It appears to me that an inconsistency is involved in this hypothesis. In a mass of apparently homogeneous substance the average distance from centre to centre of the molecule pairs will not have any directional relation, so that in the case of a longitudinal tension resisted by cohesion (say in a bar under tension) there seems no particular reason to suppose that the longitudinally cohering pairs are closer together than the lateral pairs, and yet Kelvin's hypothesis would seem to require this.

NOTE.—Sir James Dewar, in the "Encyclop. Britt.," 11th edition Vol. XVI., article "Liquid Gases," p. 756c, says :—

"COHESION.—The physical force known as cohesion is greatly increased by low temperatures. This fact is of much interest in connection with two conflicting theories of matter. Lord Kelvin's view was that the forces that hold together the ultimate particles of bodies may be accounted for without assuming any other forces than that of gravitation, or any other law than the Newtonian. An opposite view is that the phenomena of cohesion, chemical union, &c., or the general phenomena of the aggregation of molecules, depend on the molecular vibrations as a physical cause (Tolver Preston, "Physics of the Ether," p. 64). Hence, at the zero of absolute temperature, this vibrating energy being in complete abeyance, the phenomena of cohesion should cease to exist and matter generally be reduced to an incoherent heap of "cosmic dust." This second view receives no support from experiment. Atmospheric air, for instance, frozen at the temperature of liquid hydrogen, is a hard solid, the strength of which gives no hint that with a further cooling of some 20 deg. it would crumble into powder. On the contrary, the lower the scale of temperature is descended the more powerful become the forces which hold together the particles of matter."

Consider the case of a square bar 1 cm. on a side, subject to a critical tension f=2,000,000 grammes. Let the specific gravity be 8, and the number of molecules per centimetre 2×10^8.

This gives very approximately the conditions in an iron or steel bar somewhat above the elastic limit.

If we assume that the cohesion is fairly well represented by $(2\times10^8)^2$ inter-molecular links, the tension in each link is $(2\times10^6)/(4\times10^{16})=0{\cdot}5\times10^{-10}$ grammes, or say 5×10^{-8} dynes. This is, then, the tension between a pair of molecules on the two sides of an imaginary normal section. Assuming that the bond along the line connecting the two is (almost) wholly due to their mutual attractions, we have

$$5\times10^{-8}=\frac{Gm^2}{d^2},$$

where G is the gravitational constant, m is the mass of a molecule and d is the distance from centre to centre of a linked pair. G being $6{\cdot}6\times10^{-8}$, and our figures being very rough, we have approximately

$$m^2/d^2=1, \text{ so that } m=d,$$

m(grammes)$=8/(2\times10^8)^3=10^{-24}$ grammes, and $d=10^{-24}$ cm.

NOTE.—m here agrees with the mass of a hydrogen atom rather than an iron one, but this discrepancy is immaterial for our purpose.

This makes d smaller than the reputed diameter of an electron ($3{\cdot}0\times10^{-13}$ cm.), or even than a Reynolds' granule ($5{\cdot}5\times10^{-18}$ cm.), which is apparently absurd.

If the number of molecules is reduced to as low as 2×10^6 per centimetre, we have

$$\frac{2\times10^6}{4\times10^{12}}=0{\cdot}5\times10^{-6} \text{ grammes, or } 5{\cdot}0\times10^{-4} \text{ dynes.}$$

$$5{\cdot}0\times10^{-4}=\frac{6{\cdot}6\times10^{-8}\times\left(\frac{8{\cdot}0}{8\times10^{18}}\right)^2}{d^2}$$

and d=approximately 10^{-20} cm.

Even allowing for the additive effect of other molecules than those in the pair on the tension along the line between the pair it seems difficult to increase d to a reasonable figure, and in any case there seems to be no very good reason why it should differ from the reciprocal of n.

If, on the other hand, we assume that cohesion follows a law

$$t=\frac{Gm^2}{d^\gamma}$$

and d *is* equal to $1/n$, so that $t=\dfrac{Gm^2}{\left(\frac{1}{n}\right)^\gamma}$,

where t is the link between two molecules, m is the mass of a molecule, d is the inter-molecular distance, and γ is an exponent we have

$$\frac{981f}{n^2}=\frac{6{\cdot}6\times10^{-8}\times\left(\frac{8{\cdot}0}{n^3}\right)^2}{\left(\frac{1}{n}\right)^\gamma}$$

approx. $$n^4n^{-\gamma}=2\times10^{-14}$$

say $$n=10^7$$

$$10^{-7\gamma}=2\times10^{-42}.$$

Neglecting the 2, $\gamma=6$, which agrees with Boscovich's theory that the attraction at molecular distances follows a variation with a high exponent.

NOTE.—Relation of inter-molecular distance to size of molecular fields according to the Newtonian law :—

$$\frac{981f}{n^2}=\frac{G\left(\frac{\sigma}{n^3}\right)^2}{d^2}\ ;\ d=\sqrt{\frac{G\sigma^2}{981fn^4}}\quad[\sigma=\text{specific gravity}],$$

so that, other things being the same, $d\propto\dfrac{1}{n^2}$.

Relation of variational index if inter-molecular distance equals size of molecular field :—

$$\frac{981f}{n^2}=\frac{G\left(\frac{\sigma}{n^3}\right)^2}{\left(\frac{1}{n}\right)^\gamma}$$

$$\gamma\log n=\log\left[\frac{981fn^4}{G\sigma^2}\right]$$

$$\gamma=4+\frac{\log\frac{981f}{G\sigma^2}}{\log n}.$$

The above theory assumes that when the molecules are near

the limit of equilibrium the attraction (repulsion being disregarded) follows either the law

$$Gd^{-2} \text{ or } Gd^{-\gamma}.$$

The previous reasoning assumes that for the given stress between the molecules the repulsion is zero. This may not be very incorrect if the strain is such that the molecules have been dragged to the extremity of the molecular field. This condition would be indicated by an extension of not less than one-fiftieth, or in the case of simple tension, where the lateral approach of the molecules reinforces the longitudinal attractions, as much as one-quarter (for steel or iron).

Being ignorant of the arrangement of the molecules and of the laws of their repulsion and attraction, it is almost impossible to say exactly what ratio there is between the actual bond along a line between two molecules and that which there would be if these two were isolated. The ratio cannot well be less than unity, and may, perhaps, be as high as 50. Taking it as unity, let us consider two isolated molecules.

There is a repulsion between them which varies in some inverse sense with the distance from centre to centre. There is also an attraction which similarly varies, but in the position of equilibrium the two are balanced. If brought closer repulsion prevails to an extent which increases continuously with proximity. It may not become infinite, since chemical (*i.e.*, electrical) change may occur such that the atoms rearrange themselves (possibly even interpenetrate), but still closer proximity than this will experience enormous resistance from the dynamic repulsion of the extremely energetic electron systems.

A law of the form

$$t_1 = k_1 d^{-\gamma_1}$$

may serve for the repulsion where t_1 is the repulsion (negative attraction), k_1 is a coefficient, d is the inter-molecular distance, γ_1 is an index not less than unity.

The effective attraction during separation does not increase in this way, but, on the contrary, approaches a limit generally within a distance less than $2d_0$, where d_0 is the position of equilibrium. Outside this range it diminishes and becomes equal to the gravitational attraction at a distance of a few times d_0. (Various experimenters have found that cohesion is inappreciable at distances more than about 10^{-5} cm.)

If we consider the effective bond to be the resultant of an attraction and a repulsion we have

$$t=t_2-t_1,$$

and it will be found that t_2 (the molecular attraction) may also be written in a form

$$k_2 d^{-\gamma_2}$$

and agree with the experimental facts (*see* sketch).

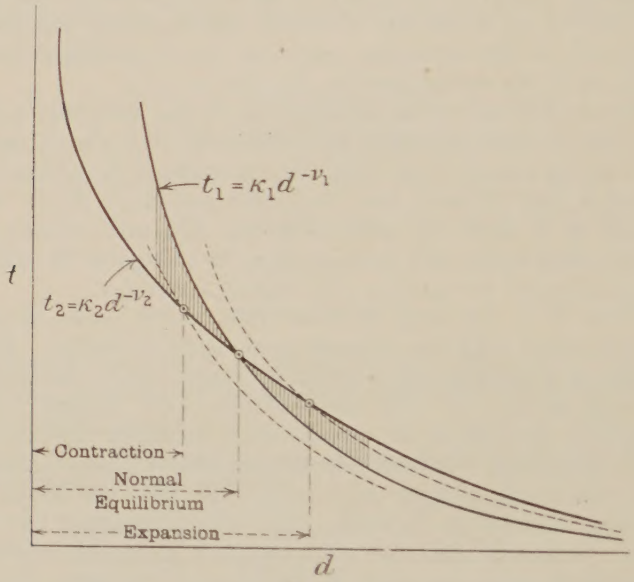


FIG. 1.

Hooke's law shows that for small increases or decreases x in the neighbourhood of d_0, t is of the form cx, where c is a constant. The conditions will be approximately satisfied if the curve representing t_1 is asymptotic to the d axis more rapidly than that representing t_2 and is asymptotic to the ordinate axis (or perhaps to one parallel thereto) less rapidly than that of t_2.

Since at equilibrium $t_1=t_2$ and d is common,

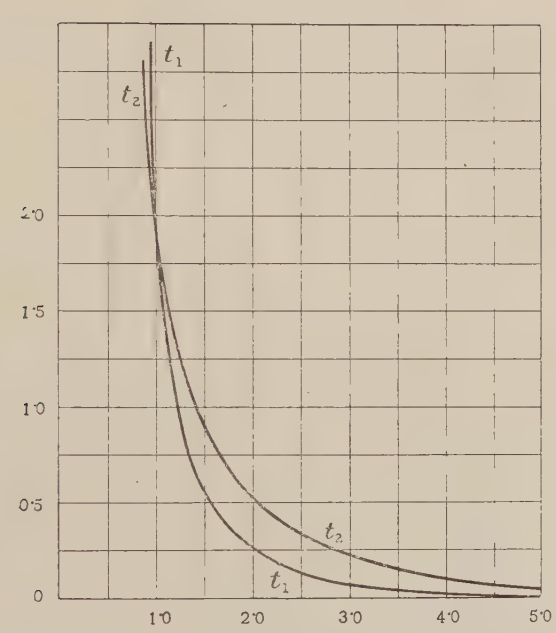
$$k_1 d_0^{-\gamma_1}=k_2 d_0^{-\gamma_2},$$

and since the slope of the t_1 curve exceeds that of the t_2 curve,

$$-\gamma_1 k_1 d^{-(\gamma_1+1)} > -\gamma_2 k_2 d^{-(\gamma_2+1)}$$

and since

$$d^{-(\gamma+1)}=d^{-\gamma} \div d$$

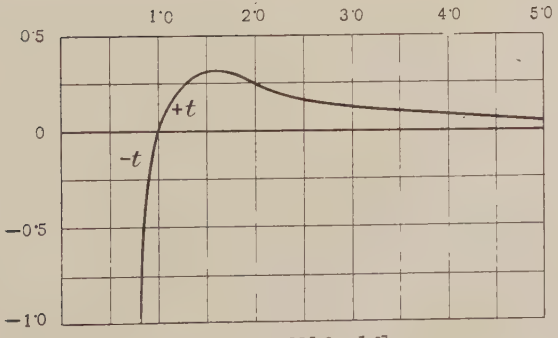


$t_1 = 2d^{-3}$.

$t_2 = 2d^{-2}$.

$\frac{dt_1}{dd} = -6d^{-4}$

$\frac{dt_2}{dd} = -4d^{-3}$



$t = t_2 - t_1 = 2[d^{-2} - d^{-3}]$.

$\frac{dt}{dd} = -2[2d^{-3} - 3d^{-4}]$.

t is a max. when $d = 1{\cdot}5$.

FIG. 2.

for the value d_0,

$$-\gamma_1(k_1 d_0^{-\gamma_1} \div d_0) > -\gamma_2(k_2 d_0^{-\gamma_2} \div d_0)$$

and $$k_1 d_0^{-\gamma_1} = k_2 d_0^{-\gamma_2}.$$

Then $$-\gamma_1 > -\gamma_2.$$

$$\therefore \quad \gamma_1 > \gamma_2.$$

The attached graphs show the curves

$$t_1 = 2d^{-3}; \quad t_2 = 2d^{-2} \text{ and } t_2 - t_1,$$

and also the curves

$$t_1 = 0{\cdot}125\,(d - 0{\cdot}5)^{-4}; \quad t_2 = 2d^{-2} \text{ and } t_2 - t_1,$$

indicating how for functions of this type there is an equilibrium value below which the first exceeds the second, and above which the second exceeds the first. It will, of course, be understood that no particular importance attaches to the indices used in these examples. In fact, the functions themselves may be of a different kind if they satisfy the required conditions.

If the coefficient k_1 is increased the equilibrium point is advanced. If decreased the equilibrium point is retracted. This corresponds to the effect of heat in causing expansion and contraction. Similarly, by sufficiently increasing k_1, the equilibrium point may be abolished, corresponding to the gaseous state.

The resultant curve in the neighbourhood of d_0 corresponds to the stress-strain curve obtained in experiments on materials. On the compression side there is an apparent discrepancy in the increase of the stress-strain ratio, but it should be observed that this curve refers to pure contraction without shear.

There does not seem to be any satisfactory method of ascertaining the values of γ_1 and γ_2, but there are several indirect ways of attacking the problem which are suggestive. The theory of gravitation and electrostatic potential suggest the index 2 for the attraction, but, as already mentioned, the coefficient "G" does not seem to agree, and if it does not agree at infinitesimal ranges there must be a change in the law at some point.

The kinetic theory of gases suggests that the repulsion varies inversely as the molecular distance.

$$\left[p \propto \frac{1}{v}; \quad v \propto d^3 \therefore p \propto d^{-3}; \quad \frac{p}{n^2} = pd^2 \propto d^{-1} \right].$$

This, however, seems to disagree with the conclusion as to $\gamma_1 > \gamma_2$ if both are more than unity.

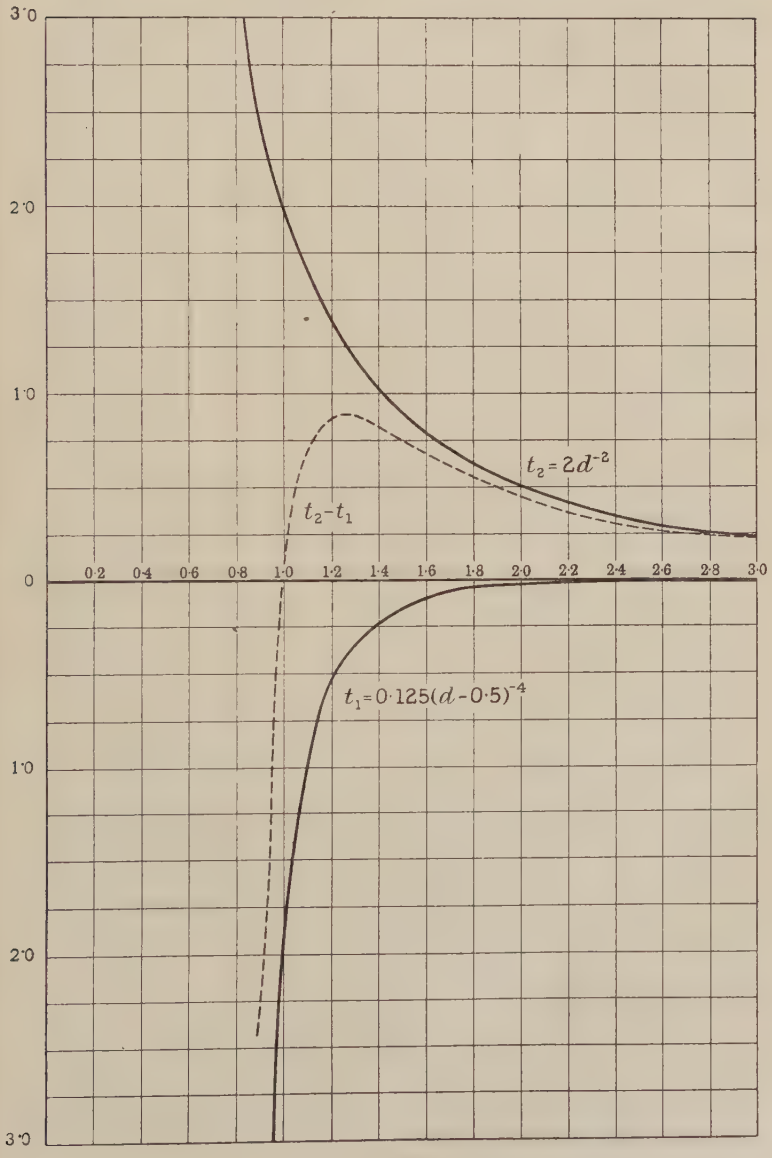


FIG. 3

The oscillation theory of repulsion is complicated by the fact that it will not simultaneously explain cohesion and temperature effects without including a potential energy store in addition to the kinetic energy. Thus, a piece of material compressed becomes heated, but when it has lost its excess of temperature by radiation and conduction it would appear that the amplitude of the oscillations had decreased, and that the frequency, after temporarily rising, had fallen to its original value. The kinetic energy of the oscillator has then decreased, but, nevertheless, the body contains strain energy if it is elastic. (Similarly, as Kelvin has shown, tension causes cooling.)

Some means of storing potential energy must be predicated in this case unless we can suppose the frequency to remain at a higher value than before straining and yet not show a perceptible temperature effect. This does not seem probable, but experiments on the combined effects of heat and stress may throw some light on the question.

Still another method is to consider the limits within which the forces are appreciable. Most metals when expanded by heat about one-fiftieth (linear), say one-sixteenth (volume), become liquid, so that for uniform dilatation of metals the net attraction (attraction-repulsion) would seem to become inappreciable when d has increased from the normal position of equilibrium to about 1·02 of that value when the temperature is raised to the melting point. Tension experiments at ordinary temperatures indicate that as much as 25 per cent., or even 50 per cent., extension can occur in the same materials before the net attraction begins to diminish; but in this case the mutual attraction, although perhaps diminished by the separation, is reinforced by the lateral approach of the particles. In the case of liquefaction it is clear that the heat greatly increases the repulsion, whereas in an isothermal tension test this does not occur. In the liquid state the molecular repulsion slightly prevails over the attraction, and is only prevented from causing dispersion (*i.e.*, vaporisation) by the atmospheric or other pressure.

The theory of elasticity in metals shows that in the absence of lateral strain a force of about 1·2 times that with lateral strain is required to produce the same longitudinal extension. Hence, it may, perhaps, be concluded that the limiting cohesion in pure dilatation is about $1 \cdot 1{\cdot}2 = 0{\cdot}83$ times the ultimate stress in uni-directional tension.

When cooled to absolute zero from usual temperatures, steel has a contraction (linear) of about 0·003. It possesses a greater cohesion at very low temperatures, a fact which may be due to the reduction of the repulsion or to increase of the attraction. The former seems more probable. In any case, it indicates that repulsion is not wholly due to the kinetic energy of temperature oscillations, or that these are not zero at the absolute zero of temperature.

Apart from the suggestions of Boscovich, Kelvin and Tolver Preston, there do not seem to have been formulated any exact theories as to cohesion, but two recent theories of gravitation are of some interest in this connection—viz., Osborne Reynolds' "granular medium" and the electronic theory of Lorenz and Crehore.

In the former ("The Sub-Mechanics of the Universe," Cambridge University Press, 1903) Reynolds finds that cohesion and gravitation can be explained by the mutual attraction of "inequalities" (*i.e.*, deficiencies) in an universally diffused granular medium of "negative mass." He says :—

"Cohesion between the singular surfaces of negative inequalities results from the terms which were not taken into account in the first approximation which correspond to gravitation. These secondary terms involve the inverse distance to the sixth power, and therefore have a very short range, and so correspond to efforts of cohesion of the singular surfaces as well as surface tensions having no effect unless the singular surfaces, or molecules, are within a distance very small compared with the diameter of the singular surface." (I., §5.)

"In the analysis for the effort of attraction of negative inequalities the ratio of the volume of the grains absent divided by the volume enclosed by the singular surface has been neglected, and it is this simplification only which renders the law of attraction—as the inverse square—the law of attraction of the singular surface at a distance.

"But this in no way limits the variation of the stresses over those portions of the space in and between the parts of the two singular surfaces which are within indefinitely small distance of each other. Such limits can only be determined by taking into account the higher terms which have been neglected.'

"This analysis I have not attempted. But it seems to me very important to notice this omission, as it appears that the attractions expressed by the higher powers of $1/r$, when the

surfaces are indefinitely near, must be of great intensity, so that, owing to sudden variations, the work done in separating the surfaces must be extremely small.

"These characteristics are those of cohesion and surface tension, and they promise to account by mechanical considerations for the hitherto obscure cohesion between the molecules as belonging to the attractions resulting from the finite diameter of the molecules divided by the curvature resulting from distortion, or we might say the complement of gravitation." (XIV., §227, slightly abridged.)

Reynolds' theory is open to objection on the ground of the apparent instability of the configurations which he deduces and seems to have been shelved, if not displaced, by the electrical theory of matter: but it is interesting to note his reference to the sixth power of the distance and the curvature effect.

Dr. Crehore has set forth an electrical theory of gravitation ("Electrical World," 1912: "Scientific American Supplement," No. 1,893, April 13, 1912, p. 238), in which, by assuming that the velocity of electrons in their orbits about the centres of atoms differs only slightly from the velocity of light,* he arrives at an expression for the mutual attraction of two atoms as follows:—

$$F=\frac{2q^2w}{K\pi v^2}\cdot\frac{e\,.\,e'}{R^2-a^2}$$

q = velocity of electron,

w = thickness of "magnetic sheet" due to the quasi-polarisation of each electron,

K = specific inductive capacity,

v = velocity of light,

R = distance from centre to centre of atoms,

a = radius of electronic orbit,

e and e' are the charges on the atoms.

By equating this to the gravitational formula, he finds that if $e=e'=1{\cdot}9\times10^{-10}$ electrostatic units and q almost exactly equals v, $w=1{\cdot}6\times10^{-36}$ cm., and that 99 per cent. of the induction (which causes the attraction by differential action

* This hypothesis does not receive confirmation from spectroscopic investigations.

at the two opposite points of the electronic orbits) is included within this thickness if $v—q=3{\cdot}9\times10^{-58}$ cm./sec.

He states that this equation is true for both great and small distances, but " It must be remembered that in deriving this formula the magnetic induction in the sheet accompanying the corpuscle was considered to be uniform throughout the volume of the positive sphere of the attracted atom. When the distance is very small the induction should be integrated throughout the volume, and the expression must not be used in case R is nearly equal to a."

If we assume that at absolute zero the molecular fields are almost in contact, the minimum value of R would seem to be $2a$ for the outside electrons. Remembering that the electrons are probably in several ring systems, $R=4a$ seems to be a fair estimate for the minimum. The expansion from absolute zero to ordinary temperatures being taken as 0·25 (an excessive value for many substances), a normal value, $R=5a$, may be appropriate for solids.

It is noteworthy that a curve of the form varying inversely as the sixth power of the distance is tangential to a curve of the form varying inversely as $R^2—a^2$ at a point $R=1{\cdot}225a$.

The differences between the curves are very small for a range of about $0{\cdot}05a$ above and below this value ; but as the value $R=1{\cdot}225a$ would imply interpenetration of the molecular fields it would seem that either the sixth power rule does not apply or that the induction correction for small values of R makes an appreciable difference.

Crehore's formula has the great advantage that it enables the gravitational constant to be retained even at the smallest distances, the variation being simply due to the second term in the denominator or the expression

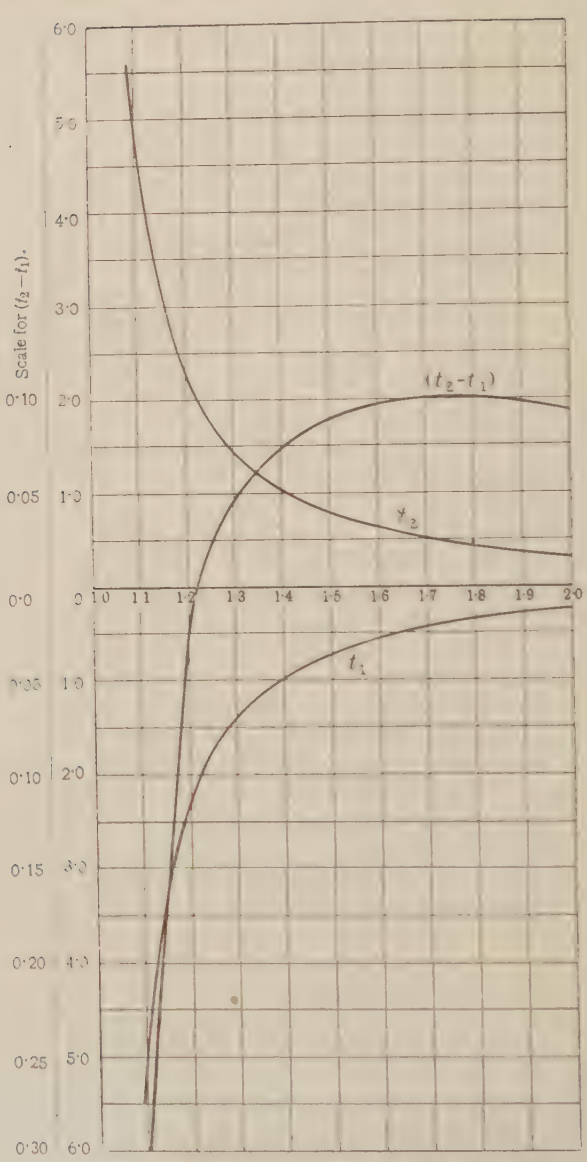
$$t_2=\frac{G\,.\,m^2}{(d^2—a^2)},$$

where t_2 is the mutual attraction, G and m are as before, and $d=R$, a being the radius of the mean effective electronic orbit.

The argument as to the indices γ previously mentioned still applies to functions of this form, so that we may, perhaps, write the repulsion

$$t_1=\frac{G_1m^2}{(d^{\gamma_1}—a^{\gamma_1})}$$

where γ_1 is greater than 2. A value of 3 for γ_1 gives a resultant



$$t_2=\frac{1}{d^2-1}$$

$$t_1=\frac{1\cdot676}{d^2-1}$$

$t_2=t_1$ when $d=1\cdot225$

FIG. 4.

curve, $t_2 - t_1$, in which the net attraction persists at a high value through a range of several times a.

Larger values of γ_1 decrease the range of the net attraction, so that experimental information on that subject indirectly leads to γ_1 if Crehore's rule is true for the attraction.

Some information might, perhaps, be expected from astronomical research. Newcomb has discussed the possibility of the index 2 in the gravitational rule being slightly inaccurate as an explanation of certain irregularities in the perturbations of the inferior planets; but Brown's exhaustive investigations of lunar theory fail to confirm this supposition. Crehore's formula would lead to the conclusion that the force is very slightly in excess of the inverse square law (which would make the index less than 2 for any particular distance), but the difference becomes inappreciable when the ratio d/a exceeds 1,000.

The Effect of Atomic Weight on the Cohesion of Solid Elements.

A scrutiny of the tensile strengths compared with atomic weights for the elementary metals shows that there is no simple relation between the two quantities. There may possibly be a periodic one, in which case it should throw light on the occurrence of gases and liquids in the series. The very existence of these at normal temperatures and pressures indicates that the conditions of infra-atomic equilibrium vary with the number of electrons (*i.e.*, the atomic weight), and it is to be presumed that the cohesion is similarly variable among the different solids. Extremely high atomic weight does not give high cohesion, nor, on the other hand, does it appear in the elements very low in the scale. There would seem to be no doubt that the high atomic weights imply a large proportion of irreducible volume or a smaller number of atoms per unit mass, and this will doubtless affect both the attraction and repulsion; but, as the effective cohesion is the difference between these, no conclusion can be drawn.

The Effect of Molecular Combination on Cohesion.

It is in this direction that the most surprising results occur. Minute variations of constitution (as in the case of the iron and copper compounds) cause great differences, and, again, the combination of two gaseous elements may produce a solid, or two solids a gas. The heat produced by or absorbed in

combination doubtless suggests an explanation, but, anyway, it is clear that the mutual chemical attraction of the atoms can immensely exceed cohesion.

The Effect of Density.

Small increases of density unaccompanied by chemical change or perceptible structural change can produce great changes in cohesion. This may partly be accounted for by considerations of invisible crystalline structure, the surfaces of the crystals being brought into more intimate contact or gaining in mutual tenacity by dimensional relations; but it would appear to be true of matter with no super-molecular structure, although the question then arises whether, apart from crystalline structure, it is possible for matter to have two densities under the same conditions of pressure, temperature and chemical composition.

The specific gravity of metals of constant chemical composition varies considerably, but it seems probable that this variation is accompanied in all cases by crystalline change. Sub-crystalline structure, although oriented, is probably of almost maximum density. Super-molecular structure can only be of maximum density if the crystals completely fit one another or are pressed together so that all parts of the surfaces of each crystal are in contact with others except on the free faces of the solid. Pressure such as this must destroy the geometrical form of the solid, and possibly these damaged angles give rise to points of relative weakness.

XXXIV. *Some Novel Laboratory Experiments.* *By* F. W. Jordan, *A.R.C.S., B.Sc.*

Received April 25, 1915.

A Vacuum Flask Condensation Calorimeter.

It occurred to me that the errors in the condensation methods of measuring the latent heat of a vapour could be almost eliminated by adopting the following method: A steady stream of the vapour is passed for a certain time through a vacuum vessel containing an empty suspended bulb, and the mass of the condensed vapour is determined. In the second experiment the same steady stream of vapour is passed for the same time through the vessel, and condensed on the same bulb filled with water. If the experiments be performed under identical conditions, the difference between the masses of condensed vapour will be that required to raise the temperature of the water to that of the vapour. Errors arising from losses of heat by convection and radiation, condensation in the tubes and wetness of the vapour can in this way be almost eliminated. The capacity of the bulb can be made large, and the masses of condensed vapour can be determined with an ordinary balance which will carry a load of 500 gm., and is sensitive to 0·01 gm. As a test of the accuracy of the method it was decided to measure the latent heat of steam.

A sectional view of the steam trap and calorimeter as designed for this purpose is shown in Fig. 1. The steam from the boiler was delivered by an upward sloping wide tube to the steam trap, where the separation of the suspended water globules from the steam was effected by centrifugal force brought into play by the rapid motion of the steam through the spiral tube A. The steam was delivered from the boiler at the rate of 14 gm. per minute, and the calculated centrifugal force of several hundred times its own weight on a water globule was considered to be sufficient to drive it to the side of the spiral tube before the escape of the steam. The steam was finally delivered to the vacuum vessel by the tube B, which was surrounded by the coils of the spiral tube. The condensed water was drained off periodically by way of the tube C.

The mouth of the thermos flask was closed by a slightly conical shallow brass cup, fitted with an outer ring of india-rubber to render it steam-tight. The steam was sprayed evenly throughout the interior of the flask by the perforated brass tube D. The exit tube E was of sufficient width to allow

of the insertion of a thermometer, and to prevent spurting of the condensed vapour from the lip of the tube. The loss of heat from the stopper was minimised by a filling of cotton-wool and a covering disc of cork. The latter prevented absorption of the steam by the cotton-wool, and was removed before weighing.

The brass bulb H was suspended by a thread looped over the

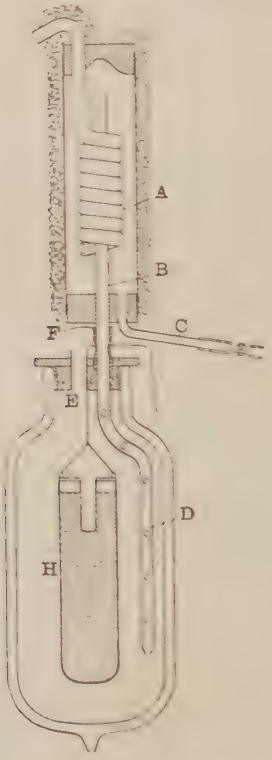


Fig 1.

inlet tube. This bulb could be filled by way of a small opening in the lid and closed by a small screw plug. The initial temperature of the bulb was obtained by lowering a thermometer through the exit tube into the re-entrant tube of the bulb.

In conducting an experiment the empty bulb was suspended inside the flask, and possible inequalities of temperature were removed by drawing a rapid current of air from the room

through the flask with a Lennox blower. The flask, together with the stoppers for the tubes, was then weighed. The thermometer was inserted into the bulb, and the initial temperature recorded after an interval of several minutes. The thermometer was then removed, and at a given instant the short indiarubber tube F was connected with the steam delivery tube. At the end of 10 minutes the supply of steam was stopped momentarily by pinching an indiarubber connecting tube; the vacuum flask removed, quickly stoppered and then weighed. It was found that this method of closing the mouth of the flask was quite satisfactory, and the weight remained perfectly steady to 0·01 gm. for many minutes. The flask was then emptied, dried and cooled by a rapid current of air to the temperature of the room. The bulb was almost filled with distilled water at room temperature, and the experiment repeated under almost identical conditions.

A preliminary experiment showed that the temperature of the full bulb attained an almost stationary value after the fifth minute, so that 10 minutes was an ample time to allow the bulb and the condensed steam to acquire the temperature of the steam. It was found as a result of a number of experiments under identical conditions that, in spite of the passage of a large amount of steam through the flask, the weight of condensed steam could be relied upon to about 0·01 gm. or 0·02 gm. The supply of steam could have been reduced after the second minute, but it was considered that the test would be more rigorous by maintaining a rapid supply of steam throughout the experiment. A reduction in the rate of supply of the vapour can be resorted to when waste of the vapour has to be avoided. If the quantity of condensed vapour is at all large it is better to catch the condensed vapour from the bulb in a shallow cup just beneath the bulb before allowing it to drip on to the bottom of the flask, where it can only be heated slowly by conduction from the upper surface. The time of passage of the vapour can then be reduced to six or seven minutes.

The thermal capacity of the flask and empty bulb may be determined to the nearest unit by measuring the steam condensed in the first minute in the calorimeter. In calculating the thermal capacity, loss of heat may be neglected, and an approximate value of the latent heat will be quite sufficient. The error arising from an omission of the correction for the thermal capacity of the vessel need not exceed 1 part in 300

in a suitably designed calorimeter, if the initial temperatures of this in the two experiments do not differ by more than a degree centigrade.

The preliminary experiments showed that as the dimensions of the apparatus were increased the final results became more concordant, and there is hardly any doubt that the latent heat of a vapour or the specific heat of a liquid, and in many cases that of a solid, could be measured to 1 part in 500.

The following dimensions of the vacuum vessel may serve as a guide to any alterations or improvements. The thermos flask was of the ordinary pint pattern; the diameter of the mouth was 2·7 cm., and the largest bulb that could be introduced had a capacity of 60 cubic cm. A thermos flask with a wider mouth and same internal volume would have given better results. The thermal capacity of the flask and empty bulb was about 23 calories. The weight of condensed steam on which the final result depended was about 9 gm. The steam was supplied at the rate of about 14 gm. per minute in most of the experiments; but this would have to be increased during the initial heating for a bulb of larger capacity. The initial temperature was read correctly to about 0·1° C. on an ordinary thermometer. The balance could carry a load of 500 gm., and the weight was measured to 0·01 gm. It may be remarked that an error of a few milligrammes in the set of weights, and this is a common fault, matters but little in this experiment.

The results obtained in six successive experiments for the latent heat of steam at 100° C. were 539·9, 539·3, 538·4, 540·1, 540 and 538·5, giving a mean value of 539·8, a result which agrees as well as can be expected with the accepted value of 540 at 100°C. in terms of the calorie at 20°C.

I have since shown, as the result of experiments with another steam calorimeter, to be described later, that the wetness of steam generated at atmospheric pressure is very small, and a separator of the ordinary pattern is quite sufficient to prevent condensed water in the delivery tubes from entering the calorimeter.

A Steam Calorimeter.

The loss of heat from the outer surface of the calorimeter can be prevented almost entirely by first raising rapidly the temperature of this surface almost to that of the steam by a sudden immersion in a current of steam, and then admitting this to the interior of the calorimeter. In order to test the accuracy of this method it was again decided to measure the latent heat of

steam and the apparatus as designed for this purpose is shown in Fig. 2.

The calorimeter A was made of hard thin sheet brass, and fitted with a screw-down lid, B. The latter was made steam-tight by a washer of moderately hard rubber. The steam when required was admitted to the interior of the calorimeter through the short inlet tube C, by removing the stopper in the exit tube D. The brass bulb E was constructed in the same way as for use with the vacuum vessel, and was suspended by a loop of thread from two small hooks on the lid of the calori-

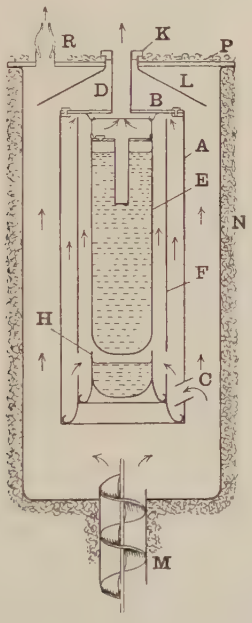


Fig. 2.—A Steam Calorimeter.

meter. The bulb E was shielded from any exchange of heat with the inner surface of the calorimeter by the radiation screen F. This consisted of a thin sheet copper vessel, perforated at the sides for the admission of steam and supported by three spring legs of thin wire on the base of the calorimeter. The upper edge of the screen was cut away to form three small projecting spikes and an alternative path for the steam. The pressure of the lid of the calorimeter against these spikes sufficed to hold the screen in position. The gain of heat by

the bulb from the radiation screen during its heating by the steam will involve a condensation of the steam on the screen, and this is included in the final weight. Thus, the introduction of this screen makes it possible to use a bulb of large thermal capacity. In the vacuum vessel the gain of heat by the bulb from the inner surface of the vessel is corrected by a condensation of the steam on this surface, and consequently this difficulty does not appear. The comparatively cool water condensed on the bulb was collected in a shallow copper cup, H, supported by three thin wire legs inside the radiation screen. The bulb was in this way almost isolated from exchange of heat with the inner surface of the calorimeter.

The calorimeter was suspended from the lid of the steam-jacket N by a circular nut of hard wood, K, screwed on to the exit tube D. A light umbrella of thin sheet copper prevented any cool-condensed water from falling on to the lid of the calorimeter. The steam before admission to the jacket was freed from suspended water by a rotary motion in the spirally divided inlet tube M. This spiral motion of the upward current of steam also contributed to a more rapid and uniform heating of the outer surface of the calorimeter and more consistent results were obtained with this device than in its absence. The additional mass of condensed steam, before the introduction of this device, did not amount to more than 0·1 gm., even when 200 gm. of steam was circulated through the calorimeter. Hence, the quantity of suspended water in the steam must have been very small, and this conclusion was still further verified by the fact that the progressive condensation in the calorimeter only amounted to a few centigrams in five minutes. Many indifferent results can be accounted for not so much by the presence of suspended water globules as by the transfer of condensed water along the surfaces of the tubes to or from the calorimeter. As an instance of this it may be noted that, during the course of an experiment on the latent heat of steam by a modified Berthelot's apparatus, it was found that 0·5 gm. of steam was condensed in the short inlet tube, whilst 24·5 gm. was condensed in the calorimeter in 14 minutes. The correction for the condensation in the inlet tube was, therefore, far from small, and under these conditions exceeded the correction for loss of heat from the calorimeter itself. In Berthelot's original experiment the effect of this condensation in the inlet tube was minimised by a rapid flow of the vapour and absorption of heat from the ring-burner.

In conducting an experiment the empty bulb E was suspended from the cover, the calorimeter was fitted together as shown in the figure, and weighed together with the stoppers for the inlet and exit tubes. The calorimeter was then attached to the lid P of the vapour jacket and a rapid stream of air was drawn through the calorimeter by a Lennox blower to equalise the temperatures of the various parts. The temperature just before immersion in the vapour jacket was taken with a thermometer inserted into the bulb E. During these operations steam was got up in the boiler and passed through the jacket for two or three minutes. The exit tube D of the calorimeter was stoppered and the lid of the vapour jacket, supported by a cord handle, was held in readiness. The temporary lid of the jacket was removed at a given instant, and the calorimeter was suddenly immersed in the steam. At the end of 20 seconds the jet of steam from the exit tube R in the lid was well developed, and this was considered to be the best instant for diverting the greater part of the steam from the outside to the inside of the calorimeter. This was effected by removing the stopper in the exit tube D of the calorimeter, and reducing the escape of steam from the exit tube R in the lid of the vapour jacket by inserting a small nozzle. The steam in every case appeared at the exit tube D of the calorimeter a few seconds after this adjustment. At the end of two minutes the supply of steam was reduced to about a third by lowering the bunsen flame. This adjustment can be conveniently and repeatedly accomplished by fastening a long arm to the handle of the gas tap, and swinging this from one stop to another. At the end of six minutes the exit tube D was stoppered ; the calorimeter was quickly withdrawn from the jacket, and its inlet tube stoppered immediately. The calorimeter was allowed to cool and then weighed. The existence of any leaky joint was rendered evident by an increase of weight, and every care was taken to avoid this defect. The rubber stoppers and washer answered perfectly well in this respect, and there was no trouble from leakage. Air was finally admitted and the calorimeter weighed.

The choice of these intervals of time was the result of a number of trial experiments. The first interval of 20 seconds was selected for the reason that an error of a second or two in its estimation affected the weight of condensed steam only to a slight extent. The next interval up to the end of the second minute was the result of an observation on the rise of temperature of the bulb when filled with about 100 cubic cm.

of water. The steam was supplied initially at the rate of about 20 gm. per minute, and the temperature of the bulb rose to about 40° C. at the end of the first minute, and 90° C. at the end of the second. The lag of the thermometer made it impossible to be precise in stating the first of these temperatures. The temperature was practically steady at the end of the fourth minute, and hence the selection of 6 minutes was considered ample to allow the bulb to acquire the temperature of the steam.

In the second experiment the bulb was nearly filled with water, and the experiment repeated under almost the same conditions. The additional amount of condensed steam was considered to be that required to raise the temperature of the water to that of the steam.

The correction to be applied for the thermal capacity of the empty bulb and radiation screen amounted only to 1 part in 1,000, when the initial temperatures of the calorimeter in the two experiments differed by 1° C. The corrections for air displacement applied to the weights affected the weights of the condensed steam and water in the bulb equally, and can, therefore, be omitted.

The following data in connection with this apparatus may serve as a guide to any improvements and alterations. The height of the brass calorimeter was 18 cm., its diameter 6·6 cm. and its weight 250 gm. The brass bulb had a capacity of about 105 cubic cm. of water, and condensed under ordinary conditions about 15 gm. of steam. The weight of the interior parts was about 100 gm., and the combined thermal capacity about 9 calories. The weight of steam condensed in the calorimeter with the empty bulb was about 1·7 gm. in six minutes, and the weights obtained in successive experiments under the same conditions agreed to about 0·02 gm.

The results obtained in seven successive experiments for the latent heat of steam at 100° C. in terms of the calories at 20° C. were 539, 539·9, 540·1, 538·8, 539, 539·6 and 539, giving a mean value of 539·3, a result which differs but little from the accepted value of 540 at 100° C. These results justify the use of this apparatus as a steam calorimeter when an accuracy of not more than 1 part in 400 is aimed at.

A Differential Telephone Receiver.

The use of this instrument for electrical measurement is not new, but it may be of some interest to know that the ordinary

receiver can be used in conjunction with a simple differentially-wound transformer to serve the same purpose.

A core, composed of a bundle of soft iron wires or a soft iron rod, is overwound differentially with two primary coils P, P, Fig. 3, and the secondary coil S, which is connected to the telephone receiver T, is arranged so that it can slide over the primary coils P, P. The advantage of this construction is that it is possible, by moving the secondary S to a particular position, to obtain almost complete silence in the receiver T, when the primary coils P are traversed oppositely by equal alternate currents. It would be difficult to wind the core of an ordinary receiver with two coils so that they were exactly differential.

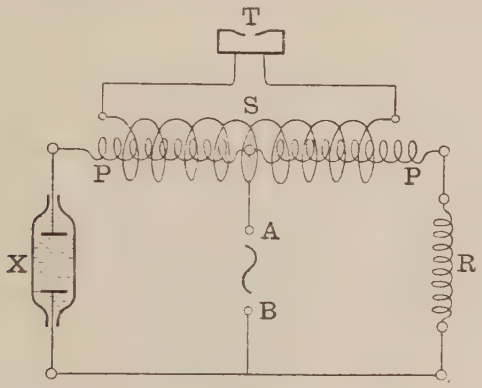


FIG. 3.—A DIFFERENTIAL TELEPHONE RECEIVER.

The resistance of an eletrolyte can be measured with the aid of this instrument by the differential arrangement shown in Fig. 3. The terminals of the secondary coil of a small induction coil are connected to A B, and the current is divided between one of the primary coils P in series with the electrolyte X and the other coil P in series with a dial box of resistance coils R. The telephone receiver will be silent when R=X, and this method is quite a sensitive one for this measurement.

The Thermal Conductivity of a Narrow Bar by a Modified Gray's Apparatus.

In Gray's apparatus the narrow bar is soldered at one end into the side of a copper boiler and at the other into a copper sphere. The heat conducted along such a narrow bar, when one end is heated by boiling water and the other cooled by contact

with the copper sphere, is so small that the temperature of the copper sphere has to be observed in some cases for 40 minutes. In order to increase the rate of transmission of heat four parallel metal bars, each about 2·5 mm. diameter and 8 cm. long, were soldered at their lower ends into a copper steam pipe A and at their upper ends into a thick copper plate B, 4 mm. thick, forming the base of the water calorimeter. The latter was chosen for two reasons, firstly, because it is more convenient and in accordance with practice, and, secondly, because an exact knowledge of the specific heat of the copper of the calori-

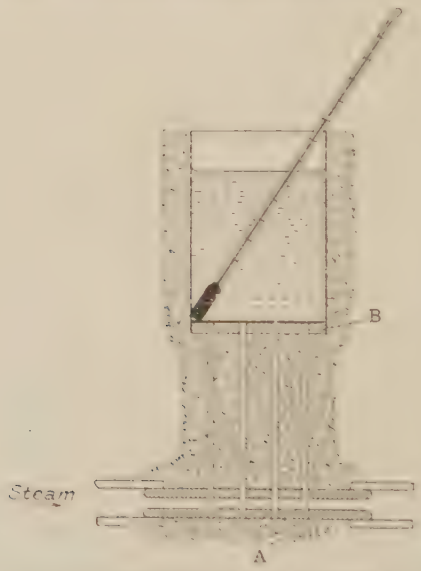


FIG. 4.—THERMAL CONDUCTIVITY OF A NARROW BAR BY A MODIFIED GRAY'S APPARATUS.

meter is not required. The use of four narrow bars, and not a single one of the same total cross section, is necessary in order to ensure a large area of transmission of heat to the water, and thus minimise the difference of temperature between the latter and the upper ends of the bars. A measurement with thermo-junctions attached to the bars showed that the error, committed in assuming that the terminal temperatures of the bars were identical with the steam and water, if this be kept stirred, is of the order of 4 per cent. The error in some experiments on thermal conductivity, arising from the use of several thermo-

meters, for the measurement of temperature gradient and conducted heat, may amount to 2 or 3 per cent. if the thermometers have not been previously compared and corrected for stem exposure. Hence, in an elementary experiment it is justifiable to make an assumption like the foregoing and reduce the measurements as much as possible. The steam pipe, bars and calorimeter were lagged with cotton wool, and exchanges of heat between the calorimeter and its surroundings can be compensated by Rumford's method.

The final expression for the thermal conductivity approximates to the ideal one that holds for steady flow conditions, and the experiment can yield almost as good a result as other experiments in the absence of refinements and troublesome corrections.

Method of Measuring Poisson's Ratio for a Rectangular Beam or Lath.

The beam is bent into an arc of uniform curvature by equal couples applied to the two ends as shown in the figure. Let

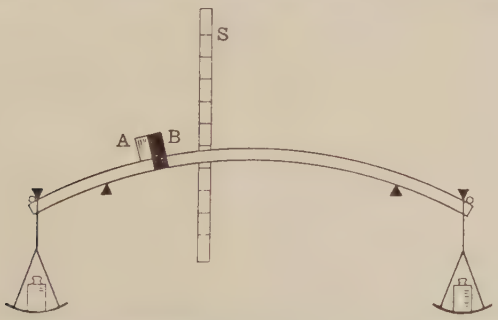


FIG. 5.—METHOD OF MEASURING POISSON'S RATIO FOR A RECTANGULAR BEAM.

R and r be the elastic and anti-elastic radii of curvature respectively. Then Poisson's ratio

$$\sigma = \frac{R}{r}.$$

The elastic radius of curvature is easily calculated from the elevation of the centre of the beam, and the distance between the knife-edge supports. To determine the anti-elastic radius of curvature I attached, at a little distance from one of the knife edges, two strips of plane mirror glass, A B, to the ver-

tical sides of the bar, so that viewed from the front of the figure their surfaces overlapped. A vertical illuminated scale was set up to one side of the common normal to the two mirrors at a distance of about a metre from the beam. A telescope was directed towards the mirror A, so that an image of the scale S formed by two reflections at the mirrors could be seen in the field of view. The anti-clastic curvature was rendered in this way quite evident and easily measurable. The mirrors were fixed near to the knife-edge support to minimise their elevations on bending the beam.

Let s be the change of scale reading on bending the beam, d the distance of mirror B from the scale, and $2b$ be the distance between the mirrors. Then the change of angle between the mirrors is $\frac{1}{2}\frac{s}{d+b}$, and the anti-clastic radius of curvature

$$=1\frac{(d+b)b}{s}.$$

If the ends of the beam be gripped in the two hands and an image of the horizontal bar of a distant window be viewed after two reflections at the mirrors A and B, then the existence of the anti-clastic curvature on bending the beam at once becomes evident by a displacement of the image. The value of Poisson's ratio for a boxwood scale obtained in this way was 0·29.

A Method of Measuring the Coincidence Period of Two Pendulums.

Each pendulum was fitted at its lower end with a platinum wire which made contact with a narrow pool of mercury at the centre of its swing. The knife-edge support and pool of one pendulum were connected electrically in series with the pool and knife-edge support of the other and with a cell and a telephone receiver. Thus, a click in the telephone was heard whenever the pendulums made contact simultaneously with the pools of mercury. The clicking in the telephone will persist over a short interval of time, which will depend on the widths of the pools of mercury and the amplitudes of the pendulums. This interval in one case was 10 seconds when the coincidence period was about 7 minutes. The coincidence period can in this way be measured correctly to one or two seconds, and the change of period of a Kater pendulum with diminishing amplitude can be readily detected. This method

appeals to students on account of its simplicity, and the only unfortunate feature of the device is that very little is left to test the skill of the student.

Achromatic Interference Fringes.

The number of visible fringes in white light produced with the aid of a Lloyd's mirror can be considerably increased by interposing a simple grating, G, of about 20 or 30 lines to the millimetre between the slit S and the Lloyd's mirror M (Fig. 6).

The mirror M consisted of a strip of good plate-glass mirror 30 cm. long and 5 cm. wide, with the silvered surface exposed to the light from the slit. The varnish at the back of this mirror was removed with turpentine and the film of yellow-scaly oxide underneath this by caustic potash. The silver surface was then washed, dried and polished with a pad of chamois leather. The vertical edges of the mirror were

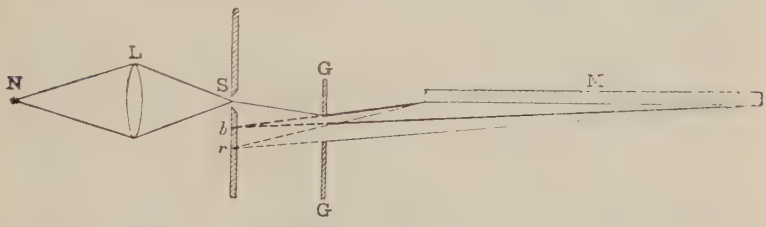


Fig. 6.—Achromatic Interference Fringes.

blackened to obstruct the transmission of light through the glass.

The grating was made by ruling the lines on a smoked-glass plate with a simple dividing engine in the following way: The anvil of a micrometer screw gauge was removed and the handle was firmly clamped to the edge of a table. The rounded end of a rod about 4 ft. long was inserted in the place of the anvil, and pressed against the screw by an elastic band. A specially pointed needle was fixed to the other end of the rod, and this was guided with a cork handle during the ruling. The greater part of the weight of the rod was supported by a counterpoise to reduce the pressure on the needle point and minimise the bending of the rod. The smoke on the glass plate was made more than usually adherent by flowing over it a weak solution of shellac in alcohol. To obtain the best results the first order spectra should be of predominant intensity and this was

effected, in accordance with theory, by making the opaque and transparent elements of the grating of nearly the same width.

The slit S was adjusted to a width of about 0·05 mm., and was illuminated by the light from a Nernst filament N. The mirror M was adjusted so that its surface was parallel to and in a line with the slit, in which position only the diffraction fringes were observable with an eye piece at a distance of about 50 cm. from the slit. The grating G had about 30 lines to the millimetre, and all but a width of a millimetre was screened from the light from the slit. The grating was arranged at a distance of about 5 cm. from the slit, with the rulings nearly parallel to the length of the slit, and just in front of the mirror M. The best width and position of the narrow grating is that in which it will just admit the first order spectrum to the mirror M. The direct white light from the grating will then fall to one side of of the mirror and interfere but little with the distinctness of the fringes. If the distance of the grating from the slit be increased to about 10 cm. the colour of the fringes can be varied without altering their width by moving the grating laterally. The lineal displacement of an element of the vertical spectrum of such a grating from the slit is very nearly proportional to its wave-length, and therefore the condition for the production of achromatic fringes is fulfilled. Interference between the lateral spectra of this grating will give the same result without the mirror M, but only if the diffracted beams overlap. For instance, in the above experiment, if the mirror M is removed, the achromatic fringes will not become visible until the width of the grating is such that, in the neighbourhood of the eye-piece, the diffracted beams overlap. Moreover, it is necessary that the lines of the grating should be more strictly parallel to the slit, whereas with Lloyd's mirror this adjustment need not be so exact.

Interference between spectra of higher order than the first will occur as the width of the grating is increased, but since these spectra are comparatively faint, the interference effects are scarcely observable, except with monochromatic light. The direct light from the grating interferes somewhat with the distinctness of the fringes, but not to the extent that one would at first imagine. The whiteness of the fringes is also impaired by an admixture of diffracted coloured pencils of light, but this adds to the colour effect and appearance of the fringes.

The direct light can be made very feeble by using a transmission grating in which the elements are of equal width, and

alternate elements are made of a thin film of gelatine of a thickness sufficient to retard the light by half the average wavelength of the most intense portion of the spectrum. I have not been able to procure such a grating, but its effect can be seen by viewing a sodium flame on a batswing burner edgewise through a small portion of the edge of a phase-reversal zone plate. The direct image of a white source of light will be coloured according to the element of the spectrum that is suppressed by the phase-reversal. The direct image will be very faint and the first order spectrum will appear almost as bright as the direct image given by a smoked-glass grating. Such a grating for the particular purpose of this experiment ought to give better results than the grating on smoked glass.

It may also be remarked that the structure of a phase-reversal grating which is almost invisible in diffused light, becomes easily visible under the conditions of this experiment owing to the appearance of the fringes formed by interference between the lateral spectra. If the slit be widened these fringes would vanish, and the grating would present almost the same appearance as in diffused light. The same remark applies to the case when the grating is viewed through a microscope.

ABSTRACT.

Condensation Calorimeters for the Measurement of Latent and Specific Heats.

(*a*) It is shown how an ordinary vacuum jacket flask can be converted into a suitable condensation calorimeter. Errors arising from loss of heat and wetness of the vapour are almost eliminated by making two experiments.

(*b*) Another condensation calorimeter is constructed on a different plan with a view to the elimination of the same errors.

The Thermal Conductivity of a Narrow Metal Bar.

Gray's apparatus for the measurement of the conductivity of a narrow bar has been modified for the purpose of rendering the loss of heat relatively small and reducing the time taken in the measurement, thus introducing the experiment to the elementary student.

The Measurement of Poisson's Ratio for a Rectangular Lath.

Two mirrors are attached to opposite sides of the bent lath for the measurement of the anti-clastic radius of curvature, and Poisson's ratio is then deduced from the observations in the usual manner.

A Method of Measuring the Coincidence Period of a Kater and a Clock Pendulum.

The Kater and the clock pendulums are electrically connected so that when at the centres of their swings a momentary current passes through a telephone receiver.

A Differential Telephone Receiver.

An ordinary receiver is connected to the secondary of a simple differentially wound transformer, and it is thus converted into a differential receiver for the purpose of electrical measurement.

Experiment on Interference Fringes.

The fringes produced by a Lloyd's mirror in white light are nearly achromatised by a simple grating on smoked glass. Other suggestive interference effects with the grating are also described.

DISCUSSION.

Mr. F. E. Smith asked why in the thermal conductivity experiment four rods were used instead of one of four times the cross-section ?

Mr. H. Moore admired the ingenuity of the pendulum experiment, but thought that to make an experiment too simple and free from corrections made it much less instructive to students. He thought the eye and ear method of observing coincidences was one in which students should be trained as much as possible.

Mr. Jordan, in reply to Mr. F. E. Smith, said that the use of four thin rods rather than one thick one was advisable, as there was a smaller temperature drop between the end of the rod and the calorimeter when the former was thin than when it was thick.

XXXV. *The Coefficient of Expansion of Sodium.* By Edgar A. Griffiths *and* Ezer Griffiths, *M.Sc.*

Received May 20, 1915.

The Coefficient of Expansion of Sodium.

Introduction.—The experiments recorded in this Paper were undertaken with the object of ascertaining the variation in the coefficient of expansion of sodium in the vicinity of its melting point.

The datum is of interest in its bearing on the modern theories of specific heat since the square of the coefficient of expansion is one of the factors which enter into the calculation of the difference between C_p, the atomic heat when the metal is free to expand and C_v, the atomic heat at constant volume.

Theoretical formulæ for the representation of C_v and its variation with temperature have been given by various writers, notably Einstein and Debye.

Einstein based his reasoning on the quantum-theory of energy and the assumption that the atoms in a solid oscillate about fixed positions, the atomic system being equivalent to a number of Planck's resonators, while Debye proceeds on the supposition that the heat vibrations depend on the elastic forces in such a way that the longest heat waves are identical with elastic vibrations.

On account of the experimental difficulties of the problem, direct determinations of C_v have not been attempted in the case of solids, and consequently to test the theoretical formulæ it is necessary to deduce C_v from a knowledge of C_p and the thermodynamic relationship * between C_p and C_v.

In the case of sodium it has been shown that C_p increases very rapidly with the temperature as the melting point is approached, rising from 6·5 at 0°C. to 7·5 at 96°C.†

Since the limiting value of C_v for an element at high temperatures is approximately 6, it seemed of interest to test the

* $C_p - C_v = a^2 evAt$.
Where a is the coefficient of cubical expansion.
e ,, ,, volumetric elasticity.
v ,, reciprocal of the density.
A ,, atomic weight.
t ,, absolute temperature.

Note.—In Phil. Trans. Roy. Soc. A. 518, p. 339, Vol. 214 (1914), the atomic weight coefficient (A) was inadvertently omitted in writing down this relationship between C_p and C_v.

† "Proc." Roy. Soc., p. 561, Vol. LXXXIX. (1914).

validity of the theoretical formulæ for a metal near its melting point by determining the data necessary for the calculation of $C_p - C_v$.

Theory of Method.

On account of its plastic nature and the tendency to oxidise, the ordinary methods for the determination of the coefficient of expansion cannot be applied to sodium.

The method described in this Paper was devised to overcome these difficulties, and the principle upon which it is based is to measure the difference in expansion of a volume of sodium and an equal volume of glass or of quartz by differential weighing under oil at various temperatures.

The essential features will be seen from an examination of Fig. 1.

Let W be the true mass of the sodium.

w ,, ,, true mass of the containing envelope.

W_1 ,, ,, apparent mass of the sodium in oil at temperature t_1.

w_1 ,, ,, apparent mass of the envelope in oil at temperature t_1.

X ,, ,, true mass of the compensating globe.

X_1 ,, ,, apparent mass in oil at temperature t_1.

Volume of sodium at temperature $t_1 = \frac{W - W_1}{d_1}$, where d_1 is the density of the oil at t_1.

If K is the coefficient of cubical expansion of sodium, volume at temperature t_2,

$$= \frac{W - W_1}{d_1}[1 + K(t_2 - t_1)].$$

If g is the coefficient of cubical expansion of the glass volume of glass envelope at temperature t_2,

$$= \frac{w - w_1}{d_1}[1 + g(t_2 - t_1)].$$

Volume of compensating globe at temperature t_2

$$= \frac{X - X_1}{d_1}[1 + g(t_2 - t_1)].$$

If Z is the additional mass required to produce equilibrium when the temperature is changed from t_1 to t_2, then Z/d_2 is the volume of oil displaced by the expansion of the sodium and its

envelope relative to the compensator, where d_2 is the density of the oil at t_2.

Hence,

$$\frac{Z}{d_2}=\frac{W-W_1}{d_1}[1+K(t_2-t_1)]+\frac{w-w_1}{d_1}[1+g(t_2-t_1)]$$
$$-\frac{X-X_1}{d_1}[1+g(t_2-t_1)].$$

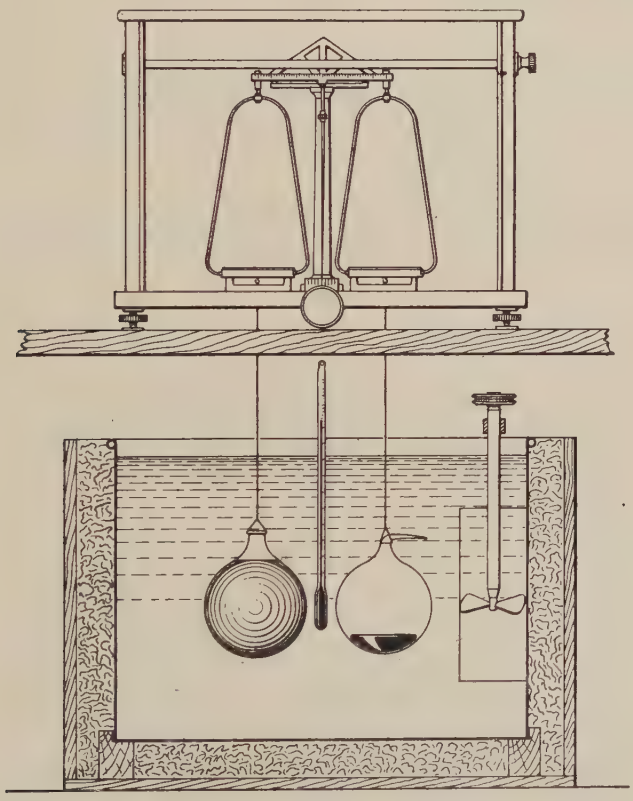


FIG. 1.

Now, by the preliminary adjustment,

$$W-W_1+w-w_1=X-X_1.$$

$$\therefore \frac{Z}{d_2}=\frac{(t_2-t_1)(W-W_1)(K-g)}{d_1}.$$

Hence,

$$K-g=\frac{Zd_1}{(t_2-t_1)(W-W_1)d_2}$$

Description of Apparatus.

In the actual observations the glass envelope containing the sodium was suspended from the left-hand pan of a 10 cm. beam balance while the sealed glass bulb was suspended from the right.

The tank contained transformer oil which could be heated to any desired temperature by means of a resistance coil, the wire being wound around the outside of the tank over a sheet of asbestos cloth.

The suspension wires were made equal, so that the two globes were symmetrically placed and close together.

Before taking observations the bath was heated to near the desired point, and the stirring continued for some time after switching off the energy supply.

The weighings were taken as soon as possible after the stirring had been stopped, to avoid the possibility of errors due to non-uniformity of temperature in the region of the globes.

The procedure adopted in setting up the apparatus was as follows :—

The glass envelope was dried and weighed together with a well-fitting cork. It was then connected by means of a short piece of wide tubing with a distilling flask containing dry sodium, and the interior thoroughly exhausted.

The sodium was heated to the molten state and, by careful tilting, poured into the envelope, the oxide being left in the larger flask.

When the sodium had solidified, the vessels were disconnected, and the envelope corked and weighed. It was then suspended in the tank, and the temperature of the oil raised past the melting point of sodium. By a little stirring of the molten metal all the gas bubbles were removed and a film of oil formed around the sodium.

When the oil had cooled down to room temperature the apparent mass of the sodium and envelope were determined when immersed in the oil and hence the volume displaced.

The volume of the compensating bulb was chosen to be equal to that of the sodium and envelope within a few cubic centimetres and then an additional small bulb was blown to bring the adjustment within a fraction of a cubic centimetre.

The final adjustment was made by fusing on minute bits of glass to the long sealing-off stem.

By a few trials it was possible to make the loss in weight in

oil of the compensator the same as that of the sodium and envelope to within very narrow limits. In the present case the loss in weight on immersion was adjusted to a residual inequality of 0·02 gm.—*i.e.*, 0·023 cubic cm., and since the displacement of the sodium was 250 cubic cm. the compensation was correct to better than one part in 10,000.

Results.

As the experiments were regarded as being merely of a preliminary nature, it is unnecessary to record all the data obtained, and the following observations are selected at random from the records :—

Weight of sodium in air	=242·868 gm.
,, ,, oil	= 29·342 ,,
,, of oil displaced	=213·526 ,,
Density of the oil at 17 deg.	= 0·854 ,,
Hence volume of the sodium	=250·03 cubic cm.

TABLE I.

Column I. Temperature of the bath.
II. Additional weight to produce equilibrium, the balance being counterpoised at 17 deg.
III. Relative increase in volume of the sodium.
IV. Increment per cubic centimetre.
V. Absolute increase of unit volume at 17 deg. after correction for the glass.

I. Temperature.	II. Equilibrium weight.	III. Relative volume increase.	IV. Increment per cubic cm.	V. Absolute increase.
25·34	0·295	0·348	0·00139	0·00161
43·11	1·057	1·261	0·00504	0·00572
61·13	1·790	2·162	0·00865	0·00979
70·90	2·158	2·625	0·01050	0·01190
91·34	3·025	3·737	0·01495	0·01688
100·21	8·625	10·728	0·04291	0·04507
76·90	2·428	2·966	0·01186	0·01342
54·36	1·505	1·810	0·00724	0·00821
23·12	0·240	0·282	0·00113	0·00129
80·65	2·560	3·137	0·01255	0·01420
96·46	3·190	3·956	0·01582	0·01789
98·12	8·530	10·590	0·04235	0·04446

Some of the observations are shown graphically in Fig 2.

The value of the coefficient of cubical expansion deduced from these observations is 0·000226 per deg., and the change in volume on passing from the solid to the liquid state amounts to 2·57 per cent.

The experiments also give the value 0·971 for the density of sodium at 17 deg., and the value 0·930 for the molten metal at its freezing point.

Calculation of $C_p - C_v$.

Taking the value $6{\cdot}5 \times 10^{10}$ for the coefficient of volumetric elasticity as determined by Richards (Jour. Chem. Soc., 1911), and calculating the difference between C_p and C_v for the

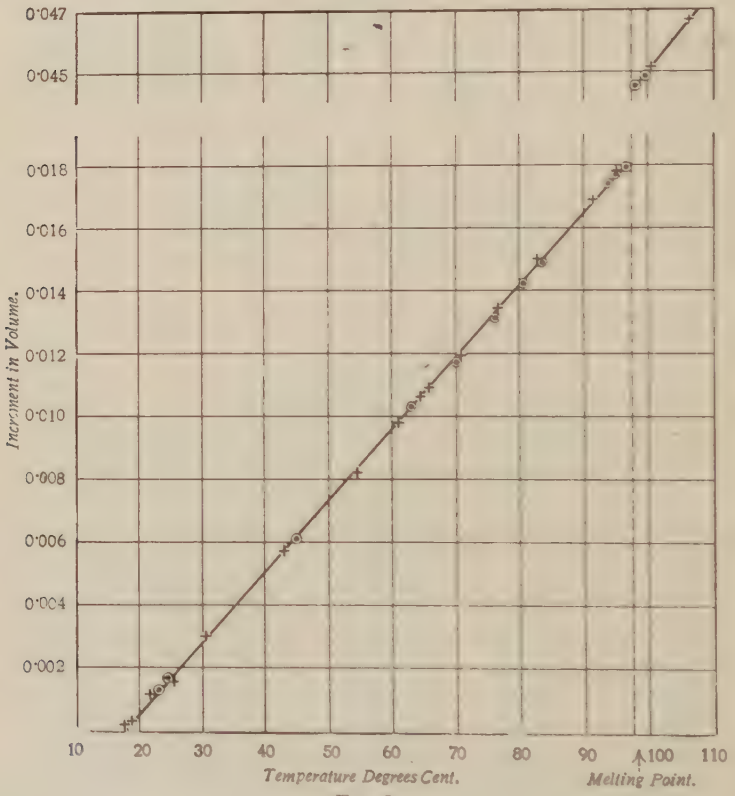


FIG. 2.

temperature of the melting point (370·6 deg. abs.), we have $C_p - C_v = 0{\cdot}70$.

This is the maximum value possible for the difference since the coefficient of volumetric elasticity probably decreases with the temperature and Richard's value was obtained at room temperature. Also the coefficient of expansion of all metals

hitherto investigated decreases very rapidly at low temperatures, and the above experiments have shown that in the vicinity of the melting point the coefficient of expansion of sodium is sensibly constant.

Hence, the value of C_v deduced from experimental data exceeds the theoretical limiting value by at least 13 per cent.

The experiments above described were intended as a preliminary series to test the method and to ascertain whether the coefficient of expansion presented any anomalies near the melting point.

In utilising the method for further work it would be advisable to use fused quartz as material for the counterpoise, since its coefficient of expansion is small and sufficiently definite to render unnecessary an independent determination of its value for the compensator.

ABSTRACT.

The thermal expansion and increase in volume on liquefaction of sodium were determined by a method based on the following principle :—

The difference in expansion of a volume of sodium and an equal volume of glass (or quartz) was measured by differential weighing under oil at various temperatures.

A volume of about 250 c.c. of sodium was suspended from one arm of a short beam balance and a weighed glass bulb of equal displacement from the other arm.

The experiments show that sodium expands quite uniformly with the temperature up to its melting point.

The value 0·000226 was deduced for the coefficient of expansion. In changing from the solid to the liquid state, an increase of 2·57 per cent. occurs in the volume.

DISCUSSION.

Mr. F. E. Smith expressed his admiration for the method employed and for the order of accuracy with which the observations appeared to have been made. Would it have been possible to obtain sufficient accuracy with the sodium in an ordinary dilatometer, the remainder of which contained oil, as was often done in the case of other metals ?

Mr. A. Campbell asked if there had been no trouble from convection in the oil at the higher temperatures. Had the authors noticed any hysteresis effect in the expansion of the glass globe such as the behaviour of a thermometer bulb would lead one to expect ?

Dr. C. Chree stated that the two curves which Mr. Griffiths had drawn on the board representing the theoretical and experimental results were very similar in form, which might indicate that the theory had something in its favour. Had the coefficient of elasticity been determined for the actual specimen used, or were general results assumed ? He believed that the elasticity varied considerably in different specimens.

Mr. J. Guild asked what precautions had been found necessary to avoid difficulties due to the very uncertain capillary effects where the suspensions enter the oil. Was the compensating bulb evacuated ? If it contained air at atmospheric pressure when cold, the pressure would

increase by about a third of an atmosphere at 100°C., which would produce an appreciable alteration in the volume of the compensator. The neglect of a correction for this pressure effect would be to give less than the true value for the expansion of the sodium, and, therefore, of $C_p - C_v$. This might account for the discrepancy between the observed and theoretical values of C_v, and it seemed advisable in future work either to evacuate the compensator or to determine its total expansion from all causes by a blank experiment.

Mr. S. D. CHALMERS asked if a check was made to see if the volume of the bulb was the same before and after the experiments. Had the authors tried if Lindemann's formula would agree with their results?

Mr. GRIFFITHS, in reply, said that in the dilatometer method the difference in expansion of sodium and oil would be measured. The coefficient of expansion of oil is about 10 times that of a metal, and would have to be determined with considerable accuracy. The method described was intended to measure the expansion relative to that of fused quartz, of which the coefficient is about 1/200 that of a metal and is accurately known. The expansion of the oil only affects the ratio of the densities at the two temperatures. The effect of convection was greatly reduced by the symmetrical disposition and closeness of the bulbs. No secular changes were found, the results obtained before and after heating to the higher temperatures agreeing (*see* Table I. and Fig. 2). With quartz bulbs no secular effects would be possible. The formulæ of Einstein and Debye were able to represent the general form of the atomic heat-temperature curve, but systematic divergencies from the experimental results occur and the theories fail to account for the abnormal increase near the melting point. Nernst and Lindemann's formulæ gave a curve very similar to Debye's (*see* "Phil. Trans." R.S., A., Vol. 214, page 343, 1914). He had assumed Richards' value for the elasticity. The suspensions were of fine tinned steel wire. As the oil wetted the surface the capillary effect would probably be constant, and the same for both wires. The compensator was heated to about 200°C. when sealing off and, being stout walled, would be little affected by the changes of pressure of the residual air when heated. The expansion of the compensator was checked by weighing it alone in water at different temperatures.

XXXVI.—*Notes on the Calculation of "Thin" Objectives.* By T. Smith, *B.A.* (*From the National Physical Laboratory.*)

Received May 27, 1915.

Synopsis.

Formulæ are given for the calculation of the curves of thin achromatic objectives, when two aberration conditions have to be satisfied. A two-lens objective satisfying the conditions as a rule cannot be cemented, because the curvatures of the inner surfaces are unequal. Formulæ are given for the calculation of a triple objective satisfying the conditions, but with only two glass-air surfaces. Numerical examples are worked out illustrating the application of the formulæ to the calculation of an astronomical object glass, and the kinds of glass are determined with which a cemented doublet satisfies the conditions for this case.

Aberration Theory of Thin Lens.

A lens system which approximates to the mathematically "thin" lens would merit considerable attention were there no uses to which it could be put but as the object glass of a telescope. Its usefulness is not, however, limited to such simple instruments, but very complex systems, such as periscopes for submarines, may be regarded as composed of a number of thin lenses. In the design of these more complicated instruments it is important to be able to arrive without excessive labour at a system which produces the desired optical effect, and these notes deal with one aspect of this problem. The positions occupied by the various lenses, their focal lengths and apertures are not here considered; they will usually be fixed by considerations of the field of view that is desired, the required magnifying power, and the mechanical limitations imposed by the dimensions of the body into which they are to be placed. Neither will the part to be taken by the several constituents in rendering possible the required perfection of the complex system be determined. It is assumed that both these problems have been solved, and that all that now remains to be done is to find how, with given kinds of glass, each subordinate system must be constructed to produce the effect that the previous investigations have shown to be necessary.

The first order aberrations of any lens system symmetrical

about an axis are determined by six quantities. In the notation used by the author these are expressed in the form of ratios, so that they are independent of the dimensions of the system. They are denoted by A, B, C, ϖ, B′, A′. When a beam of light from an infinitely distant object on the axis of the lens travelling in the positive direction is brought to the same focus by all zones of the lens, A is equal to zero. When this condition is satisfied by light from an infinitely distant object slightly off the axis of the lens, as well as from the point on the axis, B is equal to zero. A′=0 and B′=0 are the corresponding conditions when the parallel beams of light approach the lens from the negative side. The quantities C and ϖ have the same meaning for light travelling in the positive direction as for that travelling in the negative direction.

The general problem of designing a system to have arbitrarily assigned values for the six coefficients A, B, C, ϖ, B′, A′ is probably insoluble algebraically. Were such a solution obtainable it would almost certainly be too cumbersome for numerical use. When, however, the system may be treated as thin, that is to say when the thicknesses and separations of all the component lenses are so small that they may for a first approximation be neglected, the formulæ for these coefficients take simple forms, and many problems connected with thin lenses may be reduced to the solution of quadratic equations. The values for the curvatures of the lens surfaces obtained in this way are often—as in telescope objectives—in agreement to a high order of accuracy with the values obtained, when the thicknesses of the lenses are taken into account. In other cases appreciable modifications may be necessary, but much time can be saved by finding the general character of each lens before trigonometrical or other exact methods of computation are employed.

In all thin systems in which the first and last media through which the light passes are the same—for instance any lens in air—the six aberration coefficients are connected by three identical equations, viz.:

$$A+C=2B+1,\ B+B'=2C+\varpi,\ C+A'=2B'+1. \quad . \quad (1)$$

There are thus in any thin system only three degrees of freedom instead of six, as in the general case. Moreover, in practice, ϖ lies between very narrow limits, and in the majority of thin lenses it is definitely fixed when the kinds of glass to be employed are known. There are therefore only two conditions

which can in general be satisfied by a thin lens. In the numerical illustrations which follow, the conditions selected are those required in an astronomical objective, viz., that the outer zones of the lens shall have the same position for the principal focus, and have the same focal length as the central zone, or stated in technical terms, the system is to be free from central spherical aberration, and from coma for an infinitely distant object. The algebraic formulæ given below can, of course, be applied to find a system having any desired values of the aberration coefficients consistent with equation (1). The problem already mentioned of finding the properties each thin system is to have is supposed to have been solved in the form of assigning to these coefficients values consistent with these three conditions.

The coefficients by which the aberrations are measured can easily be calculated for a thin system having any number of spherical refracting surfaces. Suppose the number of these is n; let them be identified by the numerals 1, 2, 3, . . . , λ, . . . n where the numbers indicate the order in which the surfaces are met by a beam of light traversing the system in the positive direction. Let their curvatures be denoted by $R_1, R_2, R_3 \ldots, R_\lambda, \ldots R_n$, the curvature of any surface being positive if it is convex to incident light travelling in the positive direction. The powers of the surfaces will be denoted by $K_1, K_2, K_3, \ldots, K_\lambda, \ldots K_n$ with the usual convention as to sign. The refractive index of the medium between surfaces λ and $\lambda+1$ being μ_λ, we have $\mu_0=\mu_n=1$ and $K_\lambda=R_\lambda(\mu_\lambda-\mu_{\lambda-1})$. It is convenient to suppose that the focal length of the thin system considered is unity, so that

$$K_1+K_2+K_3+ \ldots +K_n=1.$$

The chromatic condition is of the form

$$0=\delta K_1+\delta K_2+\delta K_3+ \ldots +\delta K_n$$
$$=(R_1-R_2)\delta\mu_1+(R_2-R_3)\delta\mu_2+ \ldots +(R_{n-1}-R_n)\delta\mu_{n-1}.$$

If $K_{1,\lambda}$ be written for $K_1+K_2+K_3+ \ldots +K_\lambda$ the values of A, B and ω may be calculated from

$$\left.\begin{aligned} A&=1+\Sigma R_\lambda\left(K_\lambda R_\lambda-2\frac{K^2_{1,\lambda}}{\mu_\lambda}+2\frac{K^2_{1,\lambda-1}}{\mu_{\lambda-1}}\right)\\ B&=A-1+\Sigma R_\lambda\left(\frac{K_{1,\lambda}}{\mu_\lambda}-\frac{K_{1,\lambda-1}}{\mu_{\lambda-1}}\right)\\ \omega&=\Sigma\frac{K_\lambda}{\mu_{\lambda-1}\mu_\lambda}. \end{aligned}\right\} \quad . \quad . \quad (2)$$

where the summations are to be taken over all the surfaces of the system. The values of the other coefficients may be found at once from the relations (1). It should be particularly noted that ϖ is independent of the shape of any of the lenses, and is not altered by any rearrangement of them or by dividing up any lens to any extent, so long as the total power of the lenses made of each kind of glass remains the same. It may be expressed in the form

$$\varpi=\Sigma\left\{\frac{1}{\mu}(\text{sum of powers of all lenses of refractive index }\mu)\right\},$$

where the summation is taken for all kinds of glass present in the system.

Uncemented Double Objective.

For constructive purposes the formulæ for the aberration coefficients may be conveniently arranged in another way. Let it be supposed that an objective is to be constructed of two kinds of glass of refractive indices μ and μ', and that the focal length is to be the same for two colours for which the differences of refractive index are $\delta\mu$ and $\delta\mu'$.

Put $R_1=R+r$, $R_2=r$, $R_3=r'$, $R_4=r'-R'$, $\mu_0=\mu_2=\mu_4=1$, $\mu_1=\mu$, $\mu_3=\mu'$, $K_1+K_2=K$, $K_3+K_4=K'$, $K+K'=1$.

Then $K=R(\mu-1)$, $K'=R'(\mu'-1)$ and the chromatic condition is $0=R\delta\mu+R'\delta\mu'$, and therefore

$$K=\frac{\nu}{\nu-\nu'},\quad K'=\frac{-\nu'}{\nu-\nu'}$$

where $$\nu=\frac{\mu-1}{\delta\mu},\quad \nu'=\frac{\mu'-1}{\delta\mu'}.$$

Also $\varpi=\frac{K}{\mu}+\frac{K'}{\mu'}$, and all quantities are determined except r and r'. The two remaining aberration conditions are to be satisfied by a proper choice of these two quantities.

As there is frequently no a priori reason for placing the component lenses in one order rather than the reverse, it is convenient to put the formulæ in such a way that they can be as readily applied to the one case as to the other, and so far as possible avoid additional numerical work. The quantities to be expressed in terms of r and r' may therefore be C and

$(B'-B)$. The reduction is easily made from formulæ (1) and (2), with the results,

$$\left.\begin{aligned} C=&KR^2+K'R'^2+2KK'(R+R'-\varpi)\\ &+Kr\left\{3R+2K'-\frac{2}{\mu}(K-K')\right\}\\ -K'r'&\left\{3R'+2K+\frac{2}{\mu'}(K-K')\right\}\\ &+Kr^2\left(1+\frac{2}{\mu}\right)+K'r'^2\left(1+\frac{2}{\mu'}\right) \end{aligned}\right\}. \quad (3)$$

and

$$\left.\begin{aligned} B'-B=2\Big\{&KR-K'R'+K\Big(1+\frac{1}{\mu}\Big)\\ &+K'r'\Big(1+\frac{1}{\mu'}\Big)\Big\}-(K-K')\varpi \end{aligned}\right\}$$

By the second condition r' is expressed as a linear function of r, and by substituting this value in the first condition a quadratic equation is obtained for r. When this is solved the system is completely determined. As an illustration of the use of these formulæ let it be required to find the possible forms for a two lens astronomical telescope objective using the glasses for which Steinheil and Voit* carried out their investigations. The conditions to be satisfied are $C=1$, and when light approaches from the positive direction $B'-B=2+\varpi$, or from the negative direction $B'-B=-(2+\varpi)$. The values to be assigned to

$$KR-K'R'+rK\Big(1+\frac{1}{\mu}\Big)+r'K'\Big(1+\frac{1}{\mu'}\Big)$$

are thus $1+K\varpi$ and $-(1+K'\varpi)$ respectively.

The glasses employed are

$$\text{Crown}\begin{cases}\mu_D=1{\cdot}51806\\ \mu_V=1{\cdot}53356\end{cases}\qquad \text{Flint}\begin{cases}\mu_{D'}=1{\cdot}61358\\ \mu_{V'}=1{\cdot}64252\end{cases}$$

giving for the D line

$$K=2{\cdot}7348,\ K'=-1{\cdot}7348,\ R=5{\cdot}2789,\ R'=-2{\cdot}8273,$$

$$\varpi=\frac{K}{\mu}+\frac{K'}{\mu'}=1{\cdot}8015-1{\cdot}0751=0{\cdot}7264.$$

* "Handbuch der Angewandten Optik."

The condition C=1 gives

$$6{\cdot}3378r^2-3{\cdot}8851r'^2+17{\cdot}7177r+4{\cdot}3848r'+44{\cdot}9734=0.$$

B′−B=2+ϖ gives

$$4{\cdot}5363r-2{\cdot}8099r'+6{\cdot}5454=0,$$

and B′−B=−(2+ϖ) gives

$$4{\cdot}5363r-2{\cdot}8099r'+9{\cdot}2718=0.$$

The solutions are

$$\left.\begin{matrix} r=-3{\cdot}6410 \\ r'=-3{\cdot}5486 \end{matrix}\right\}\quad \left.\begin{matrix} r=+2{\cdot}4732 \\ r'=+6{\cdot}3221 \end{matrix}\right\}\quad \left.\begin{matrix} r=-5{\cdot}2443 \\ r'=-5{\cdot}1666 \end{matrix}\right\} \text{ and } \left.\begin{matrix} r=+0{\cdot}8630 \\ r'=-4{\cdot}6929 \end{matrix}\right\}$$

the first two for parallel light incident on the crown lens, and the last two for parallel light incident on the flint lens. These forms are illustrated in Fig. 1 on the same scale as Steinheil &

Fig. 1.

Voit's figures. Of these four forms the second and fourth are probably never used. It usually happens when alternative forms are available that the one involving the least curvatures is adopted, the probability being that with this construction a larger aperture can be used than with the other forms before the higher order aberrations become perceptible.

In the more desirable forms the curvatures of the inner surfaces are nearly, but not quite, equal to one another. This result is true of nearly all the glasses likely to be used in telescope construction. It is often a decided advantage to employ an objective in which these surfaces are of equal curvature and are cemented together. A very appreciable gain in the amount of light transmitted is secured by eliminating glass air surfaces in this way. In some cases it is not necessary to satisfy the assumed conditions to a high order of

accuracy, and mathematical exactitude in the removal of aberrations may be sacrificed to gain increased light transmitting qualities. It might be supposed, for example in the first solution obtained above, that the effect of slightly altering the shape of one or both of the lenses to make their curvatures alike, would be of little consequence, except perhaps in the case of a large objective, the difference in curvature being too small to be perceptible from the diagram. If we perform the calculations putting firstly $r=r'=-3{\cdot}6410$ and secondly $r=r'=-3{\cdot}5486$ we find that the aberration coefficients have the values $A=-3{\cdot}5052$, $B=-3{\cdot}2456$ and $A=-3{\cdot}4114$, $B=-2{\cdot}9923$ respectively. It will be observed that the signs of the errors introduced are the same, whether the crown lens is bent towards the flint or the flint towards the crown. An idea of the magnitude of the errors to which the above numbers correspond may be formed by comparing them with the values for a single thin lens of refractive index 1·5. If such a lens is planoconvex with the convex side towards the incident light, its coefficients will have the values $A=2\frac{1}{3}$, $B=2\frac{2}{3}$; if it is equiconvex $A=3\frac{1}{3}$, $B=2$; if planoconvex with the plane side towards the incident light $A=9$, $B=6$. Thus the spherical aberration introduced by making the combination cemented with the minimum change of shape is about equal, but opposite in sign, to that of the equiconvex lens, and the coma (measured by $A-B$) is approximately equal and opposite in sign to that of the planoconvex lens with the curved surface turned towards the incident light.

When central spherical aberration is present, rays incident parallel to the axis on the zone of diameter d will meet the axis at a distance $Ad^2/8f$ nearer to the lens than the principal focus for paraxial rays, where f is the focal length of the lens. The diameter of the so-called circle of confusion in the paraxial image plane is $Ad^3/8f^2$. Coma has been described as a zonal variation of the focal length, but this description is not strictly accurate, inasmuch as when it is present all rays from the same zone do not meet the image plane in one point. Rays from diametrically opposite points of any one zone meet the image plane in one and the same point, and the fact that both rays in an axial or primary plane do so agree is probably responsible for this way of expressing the meaning of coma. Rays in a primary plane meet the paraxial image plane in the points determined by assuming the focal length to be $f+3(A-B)d^2/8f$, and rays incident in a secondary plane as

though the focal length were $f+(A-Bd^2/5f)$. Rays from other parts of the zone meet the image plane in the circle having the image points for rays in the primary and secondary planes as the extremities of a diameter. When coma and central spherical aberration are both present their effects on each ray are additive. In the lens taken as an illustration, if the aperture were one-tenth the focal length, the longitudinal spherical aberration introduced by modifying the lens to the extent necessary to enable the crown and flint glasses to be cemented together, would be nearly $\frac{1}{2}$ per cent. of the focal length, and the coma would correspond to a decrease in the focal length for the outer zones amounting to about one-eighth of 1 per cent. For many purposes such errors would be quite inadmissible. Before turning to systems which satisfy the conditions, and at the same time have only two glass air surfaces, we may note one or two cemented lenses of the type so far considered. With the above glasses, when there is no coma with the crown lens leading it is found that

$$r=r'=-3{\cdot}7914, \quad A=-3{\cdot}5684,$$

and with the flint lens leading

$$r=r'=-5{\cdot}3707, \quad A'=-2{\cdot}9842.$$

When spherical aberration is corrected the solutions are

$$\left.\begin{aligned} r=r'&=-5{\cdot}0080\\ B&=-2{\cdot}1003 \end{aligned}\right\} \text{ and } \left\{\begin{aligned} r=r'&=-2{\cdot}5956\\ B&=+2{\cdot}0645 \end{aligned}\right.$$

with the crown lens leading, and

$$\left.\begin{aligned} r=r'&=-6{\cdot}3242\\ B'&=+1{\cdot}6462 \end{aligned}\right\} \text{ and } \left\{\begin{aligned} r=r'&=-4{\cdot}0947\\ B'&=-2{\cdot}2027 \end{aligned}\right.$$

with the flint lens leading. Of these four solutions the second has the crown lens very approximately equiconvex, and the flint plano concave: this form is frequently adopted for small objectives on account of its small manufacturing cost.

Effect of Bending Thin Objectives.

When a lens system must have all its inner surfaces cemented together, only one degree of freedom is present apart from those obtainable by changes in the kinds of glass used, in the number

of component lenses and in the order in which they are arranged on the axis. In a thin lens this freedom is exercised by increasing the curvature of each refracting surface by the same amount. On substituting in equations (2) $r+R_\lambda$ for R_λ, $K_\lambda+r(\mu_\lambda-\mu_{\lambda-1})$ for K_λ, and $K_{1,\lambda}+r(\mu_\lambda-1)$ for $K_{1,\lambda}$, where r is

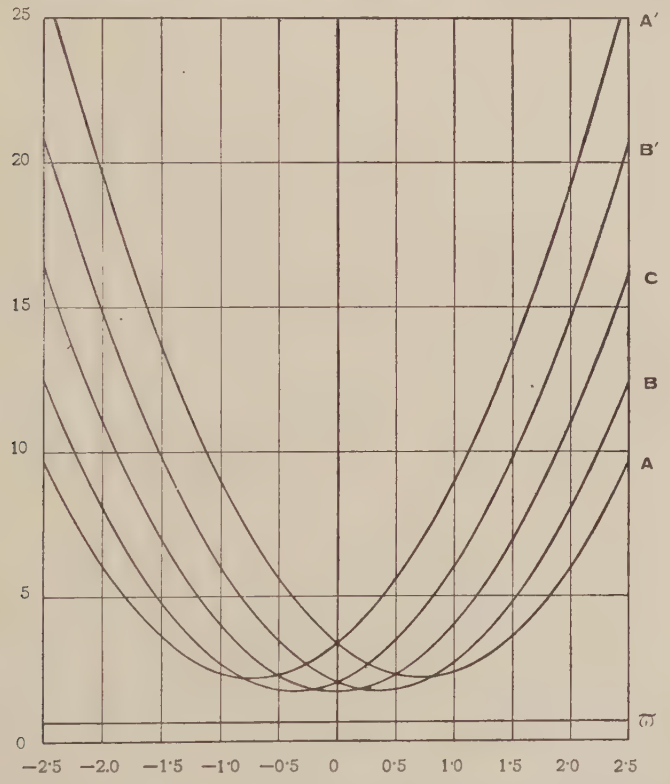


FIG. 2.

the increment of curvature of each surface of the system it will be found that

$$\left.\begin{aligned} C&=C_0+r(2B_0'-2B_0-\Sigma K_\lambda R_\lambda)+r^2(1+2\varpi).\\ \text{and}\quad B'-B&=B_0'-B_0+2r(1+\varpi)\end{aligned}\right\}, \quad . \quad (4)$$

where the suffix $_0$ is attached to the values of the aberration coefficients for $r=0$. Thus when these coefficients are known for any thin system, the effect of any bending of the system as a whole is determinable at once when $\Sigma K_\lambda R_\lambda$ has been found

for the unbent system. Each product contributing to this sum has already been evaluated in the course of finding the aberration coefficients by equations (2).

The standard form for the equations expressing the effect of

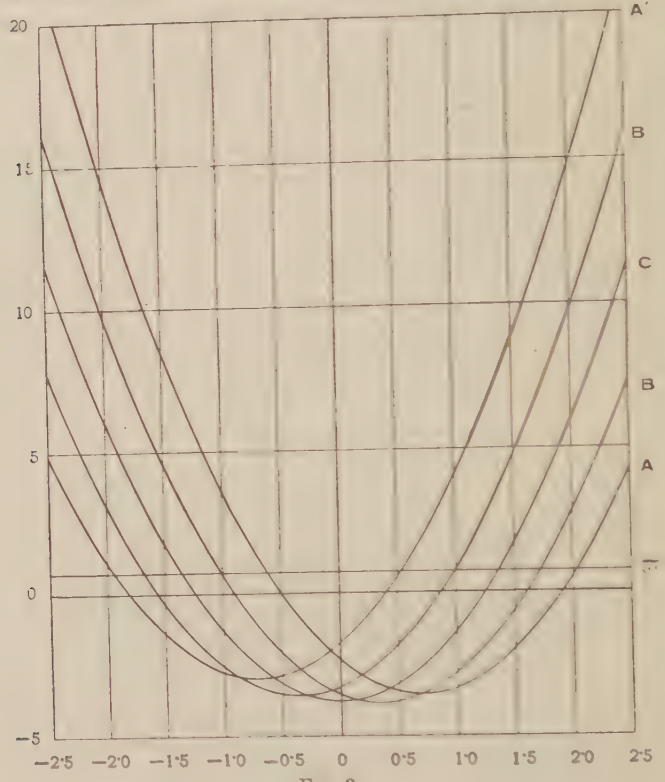


Fig. 3.

bending may be taken as that for which $r=0$ when C is a minimum. The equations then become

$$\begin{aligned}
A &= A_0 - 2r(1+\varpi) + r^2(1+2\varpi) \\
B &= B_0 - r(1+\varpi) + r^2(1+2\varpi) \\
C &= C_0 + r^2(1+2\varpi) \\
B' &= B_0' + r(1+\varpi) + r^2(1+2\varpi) \\
A' &= A_0' + 2r(1+\varpi) + r^2(1+2\varpi)
\end{aligned}$$

The curves expressing the variation of the coefficients with r as independent variable, are thus equal parabolas with their

axes vertical and separated horizontally at equal intervals of $(1+2\varpi)/2(1+\varpi)$. Fig. 2 shows these curves for a single lens of refractive index 1·5. In Fig. 3 the curves for cemented doublets made of the glasses used in the foregoing numerical illustrations are given. In the case of the single lens the curves lie wholly on the positive side of the axis of abscissæ. In the other case the curves cross this axis. The cemented lens is overcorrected for a distant object in that the intersection of curves A and B lies on the negative side of the axis. It is evident from the fact that a cemented doublet is overcorrected when made of the usual glasses, but is undercorrected when the refractive index has the same value throughout the lens, that it is theoretically possible to choose two glasses such that the intersection of the curves A and B will lie on the axis when the system is corrected for colour and is also cemented. The practical realisation of these conditions depends upon our ability to make glasses of the required relative dispersions, having regard to the values of the refractive indices. This question is dealt with later; meanwhile it is worth considering how the conditions can be fulfilled for a cemented achromatic objective when it has to be made of two given kinds of glass.

Triple Cemented Lenses.

The simplest way of attempting this is to assume that two lenses will be made of one kind of glass, and that the lens of the other kind will be cemented between these two. Such a system can be regarded as built up of two cemented achromatic doublets placed back to back. Now the aberration coefficients of a compound system composed of two similar systems placed back to back are easily expressed in terms of those of the components. Let large capitals denote the aberration coefficients of the compound system, and small capitals those of the simple systems. If the compound system of power unity be made up of simple systems of powers $1-\theta$ and θ, the relations

$$\mathrm{C}=\mathrm{C}+\theta(1-\theta)(\mathrm{A}-4\mathrm{B}-1)$$

and

$$\mathrm{B}'-\mathrm{B}=(1-2\theta)(\mathrm{A}-4\mathrm{B}-1+3\mathrm{C}+\varpi)$$

hold whether the simple systems are thin or not. When the

simple systems are thin these relations reduce, in virtue of equations (1) to

$$\left.\begin{aligned} \mathbf{C} &= \mathrm{C} - \theta(1-\theta)(2\mathrm{B}-\mathrm{C}) = \mathrm{C} - \theta(1-\theta)(3\mathrm{C}+\sigma-\mathrm{B}'-\mathrm{B}) \\ \text{and} \qquad \mathbf{B}'-\mathbf{B} &= (1-2\theta)(\mathrm{B}'-\mathrm{B}) \end{aligned}\right\} \quad (5)^*$$

Let the value of $\Sigma \mathrm{K}_\lambda \mathrm{R}_\lambda$ for the simple system be denoted by σ: then for the compound system $\Sigma \mathrm{K}_\lambda \mathrm{R}_\lambda = (1-2\theta)\sigma$. Combining with equations (5) the resulting change in the coefficients due to a change of curvature r in each surface from equations (4) we obtain the generalised formulæ

$$\left.\begin{aligned} \mathbf{C} &= \mathrm{C} - \theta(1-\theta)(2\mathrm{B}+\mathrm{C}) + r(1-2\theta)(2\mathrm{B}'-2\mathrm{B}-\sigma) + r^2(1-2\theta) \\ \mathbf{B}'-\mathbf{B} &= (1-2\theta)(\mathrm{B}'-\mathrm{B}) + 2r(1-\theta) \end{aligned}\right\} \quad (6)$$

giving the effect of combining two systems together and bending the combined system to an arbitrary extent. In order that the coefficients may be those for a system with only two glass air surfaces it is only necessary that the subsystems, in addition to having their inner surfaces cemented together, should have their last surfaces plane in the form to which the values of C, B′−B, and σ used on the right of equations (6) refer.

The form of these equations shows that the general problem of satisfying two aberration conditions is resolved into the solution of a quadratic equation. Thus apart from the occurrence of imaginary roots—four thin cemented achromatic triplet objectives can be made with two given kinds of glass to satisfy two arbitrary aberration conditions, two of them having one kind of glass enclosed between lenses of the other kind, and two with this glass enclosing a lens of the second type of glass. Since the triplet form provides solutions of the general problem we have been considering there is nothing to be gained by the consideration of thin systems of a more complex type. With the Steinheil & Voit glasses, it is readily found that a double

* These results may of course be verified by substituting in formulæ (2) for the powers, curvatures and refractive indices the following series of values :—

$$(1-\theta)\mathrm{K}_1, (1-\theta)\mathrm{K}_2, (1-\theta)\mathrm{K}_3, \ldots (1-\theta)\mathrm{K}_n, \theta\mathrm{K}_2, \theta\mathrm{K}_{n-1}, \theta\mathrm{K}_{n-2}, \ldots \theta\mathrm{K}_2, \theta\mathrm{K}_1$$

$$(1-\theta)\mathrm{R}_1, (1-\theta)\mathrm{R}_2, (1-\theta)\mathrm{R}_3, \ldots (1-\theta)\mathrm{R}_n, -\theta\mathrm{R}_n, -\theta\mathrm{R}_{n-1}, -\theta\mathrm{R}_{n-2}, \ldots -\theta\mathrm{R}_2, -\theta\mathrm{R}_1$$

$$1, \mu_1, \mu_2, \mu_3, \ldots \mu_{n-1}, 1, \mu_{n-1}, \mu_{n-2}, \ldots \mu_2, \mu_1, 1.$$

cemented objective with the second surface of the flint lens plane has

$$C=3{\cdot}0890,\ B'-B=6{\cdot}0551,\ \Sigma K_\lambda R_\lambda=3{\cdot}8773,$$

and therefore for a cemented triplet

$$C=2{\cdot}4528r^2+8{\cdot}2329r(1-2\theta)-3{\cdot}9383\theta(1-\theta)+3{\cdot}0890$$

$$B'-B=3{\cdot}4528r+6{\cdot}0551(1-2\theta),$$

the solutions of which when $C=1$ and $B'-B=2+\varpi$ are

$$\left.\begin{matrix}\theta=0{\cdot}1783\\ r=-0{\cdot}3385\end{matrix}\right\} \text{ and } \left.\begin{matrix}\theta=0{\cdot}8464\\ r=2{\cdot}0045\end{matrix}\right\},$$

giving for the curvatures of the four surfaces

$$\left.\begin{matrix}1{\cdot}6758\\ -2{\cdot}6616\\ 0{\cdot}1657\\ -0{\cdot}7758\end{matrix}\right\} \text{ and } \left.\begin{matrix}2{\cdot}3811\\ 1{\cdot}5701\\ 4{\cdot}3974\\ -0{\cdot}0705\end{matrix}\right\} \text{ respectively.}$$

In the doublet with flint lens leading and last surface plane $C=-2{\cdot}3528$, $B'-B=2{\cdot}4097$, $\Sigma K_\lambda R_\lambda=1{\cdot}0259$, leading to equations for this form of cemented triplet

$$C=2{\cdot}4528r^2+3{\cdot}7935r(1-2\theta)+8{\cdot}7417\theta(1-\theta)-2{\cdot}3528$$

$$B'-B=3{\cdot}4528r+2{\cdot}4097(1-2\theta)$$

and to the solutions

$$\left.\begin{matrix}\theta=0{\cdot}6389\\ r=0{\cdot}9835\end{matrix}\right\} \text{ and } \left.\begin{matrix}\theta=0{\cdot}3210\\ r=0{\cdot}5397\end{matrix}\right\},$$

with curvatures

$$\left.\begin{matrix}1{\cdot}8688\\ 2{\cdot}8897\\ -2{\cdot}3892\\ -0{\cdot}5828\end{matrix}\right\} \text{ and } \left.\begin{matrix}2{\cdot}2044\\ 4{\cdot}1244\\ -1{\cdot}1546\\ -0{\cdot}2471\end{matrix}\right\} \text{ respectively.}$$

As was to be expected the curvatures compare favourably with those for the double forms of objective satisfying the same conditions, viz. :—

$$\left.\begin{matrix}1{\cdot}6379\\ -3{\cdot}6410\\ -3{\cdot}5486\\ -0{\cdot}7213\end{matrix}\right\}, \left.\begin{matrix}7{\cdot}7521\\ 2{\cdot}4732\\ 6{\cdot}3221\\ 9{\cdot}1494\end{matrix}\right\}, \left.\begin{matrix}2{\cdot}3393\\ 5{\cdot}1666\\ 5{\cdot}2443\\ -0{\cdot}0346\end{matrix}\right\}, \left.\begin{matrix}-7{\cdot}5202\\ -4{\cdot}6929\\ -0{\cdot}8630\\ -4{\cdot}4159\end{matrix}\right\}.$$

It will be noted that one of the forms with crown glass outside is slightly better than the more favourable form with flint outside, as regards smallness of curvature, and that the

Fig. 4.

other two forms are not as satisfactory in this respect as the best doublet form. The four triplets are illustrated in Fig. 4.

Cemented Double Objectives for Telescopes.*

From the point of view of reduced cost of manufacture a doublet lens is preferable to a triplet, and it is of interest to see what properties the glasses to be used must have if a cemented objective is to be free from colour and from spherical aberration and coma. Assuming a refractive index for the flint lens of 1·6, the ratio of the dispersions of crown and flint glasses for various refractive indices of the crown lens is shown by the curve of Fig. 5. This ratio demands less dispersive power from the crown lens than is found in the ordinary glasses; even when the r for the flint lens is as low as is obtainable with a refractive index of 1·6, the only glasses which satisfy the conditions are phosphate crowns, which, on account of their instability, are out of the question, and medium baryta crowns

* Cemented doublets free from spherical aberration and coma for a distant object have been very thoroughly investigated by Harting, who has prepared tables giving the necessary curvatures for the surfaces and the relative dispersions of the glasses for given refractive indices, and has further considered what glass combinations satisfy the conditions. His list of possible combinations does not, of course, include the thor crown combination mentioned above, as this glass has been introduced since Hartings' investigation was carried out, and his tables are not sufficiently extended to include such glasses. Von Hoegh has given equations for the determination of such doublets when the object is at a given finite distance from the lens. The diagrams given above were obtained by a method which avoids the solution of a quintic equation. An account of the work of both Harting and Von Hoegh on this subject will be found in Gleichen's "Lehrbuch der Geometrischen Optik," p. 323.

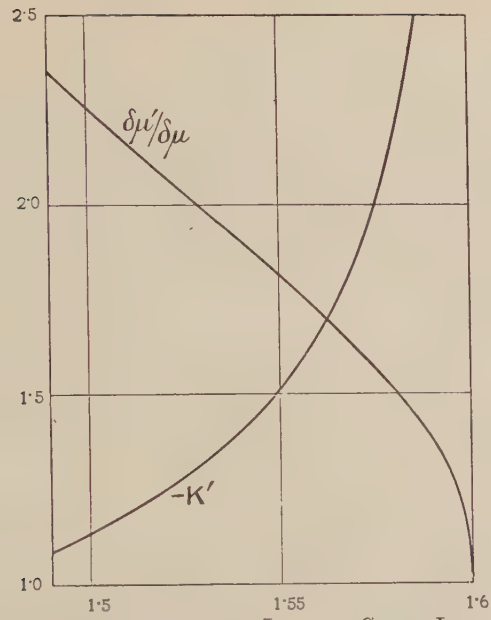


FIG. 5.—REFRACTIVE INDEX OF CROWN LENS.

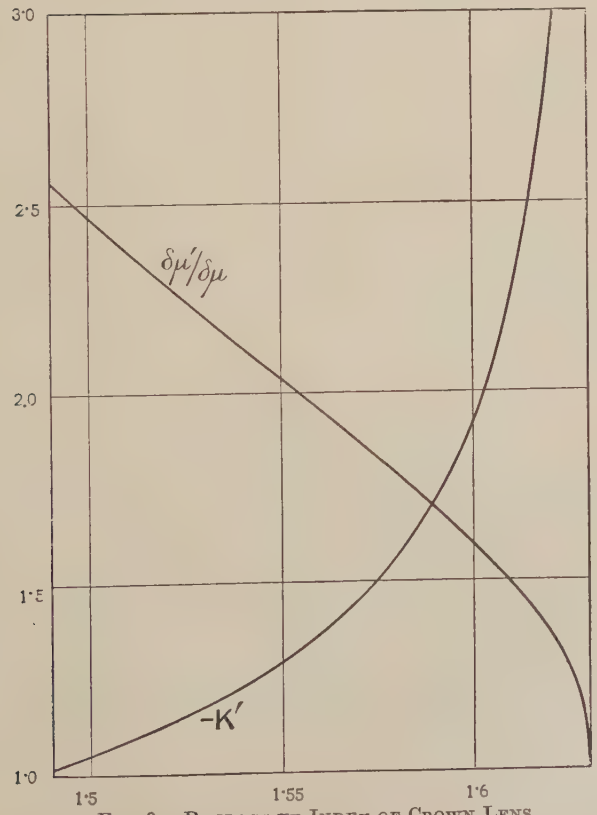


FIG. 6.—REFRACTIVE INDEX OF CROWN LENS.

which require greater curvatures than is desirable. If the refractive index of the flint lens is increased to 1·63 the curve obtained is shown in Fig. 6. Here the medium baryta crowns are again on the curve, but this now passes through the position of one of the fluor crowns, which have been recently introduced by Messrs. Schott. So far as the published descriptions state, this glass appears to be in every way a desirable one to employ in a telescope objective. Its price is, however, much higher than that of the more familiar glasses. It appears undesirable to employ a flint of greater refractive index than 1·63, as such glasses are coloured and cannot be handled without risk of tarnishing.

ABSTRACT.

Lens systems which are symmetrical about an axis have in general six degrees of freedom for first order aberrations. Thin systems have only three degrees of freedom, and in consequence of the limited range of glasses only two degrees of freedom are practically available. In achromatic combinations of two lenses these two degrees of freedom are controlled by the general shape as distinct from the total power of each lens. In general when two given conditions are satisfied the curvatures of the inner surfaces are not equal, so that a cemented combination of two lenses is not possible. Owing to the increased light transmitting powers it is often necessary to have only two glass air surfaces, and thus more than two component lenses are necessary. The effect of bending any thin system as a whole by increasing the curvature of each surface by the same amount is investigated, and it is shown that with two given kinds of glass a triple cemented lens can be formed satisfying two arbitrary aberration conditions. The problem reduces to the solution of quadratic equations, and in general there are four solutions. Illustrations are given of astronomical objectives of both double uncemented and triple cemented forms, and the glasses are determined for which a doublet can be cemented.

DISCUSSION.

Mr. S. D. Chalmers expressed his appreciation of the Paper. One could get a rapid survey from it of the results which were possible. The special problem of a lens corrected for spherical aberration and coma had been discussed by Harting and also by Von Hoegh. The latter had solved the problem on lines which seemed to him rather simpler to use. He did not quite agree with the author's statement that the use of phosphate crowns had been entirely discontinued. Glasses very like these have been used fairly frequently, especially by Continental makers. There were cases where in computing a lens it was not desired to achromatise completely; where, for example, after correcting for spherical aberration and coma, some residual chromatic aberration was desirable to neutralise the chromatic aberration of the prisms in binoculars.

Mr. A. Campbell asked if the small thickness of the cement used in cemented lenses made an appreciable difference, and if its refractive

properties entered into the calculations. What was the range of refractive index available in the modern optical glasses ?

The Author, in reply, was surprised to hear that phosphate glasses were still employed in any case in which they could possibly be avoided. The cement used was always Canada balsam and was entirely neglected in all computations. A range of refractive index of 1·47 to 1·79 was available, but the extreme values were rarely used, 1·49 to about 1·65 or so being the range usually utilised.

XXXVII.—*On Tracing Rays Through an Optical System.* By T. Smith, *B.A.* (*From the National Physical Laboratory.*)

Received May 27, 1915.

The most troublesome calculations which have to be made in computing an optical system are those relating to rays not lying in an axial plane. The methods hitherto used are trigonometrical. Perhaps the formulæ most extensively employed are those of Von Seidel. A complete account of them will be found in an appendix to Steinheil & Voit's "Handbuch der Angewandten Optik." They may, with minor changes in the notation, be summarised as follows.

Let OP and O′P be the incident and refracted rays for a

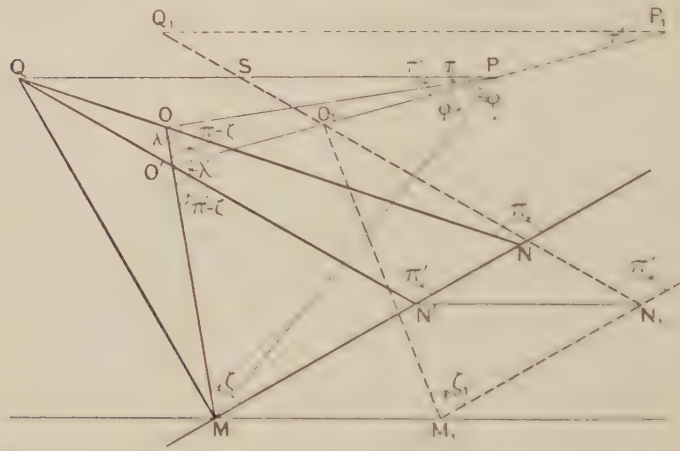


Diagram illustrating Von Seidel's Method of Tracing a General Ray through a Lens System.

spherical surface whose centre is at M. Take O and O′ in the plane through M normal to the axis of the system, and let Q be the foot of the perpendicular let fall from the point of refraction P to this plane. In the diagram, lines in this plane are drawn as continuous thick lines, and lines not in a plane normal to the axis are shown thin. The interrupted lines refer to refraction at a subsequent surface. The angles of incidence and emergence are φ and φ': the angles made by the incident and refracted rays with the axis are τ, τ'. The

angles made by QO and QO′ with an arbitrary direction MN′N in the normal plane are π and π'. The points M, O′ and O lie on a straight line inclined at an angle ζ with MN, so that the angles MON and MO′N′ are $\pi-\zeta$ and $\pi'-\zeta$ respectively. Let the angles MOP and MO′P be called λ and λ', the lengths MO and MO′ U and U′, and the radius of the sphere R. Then the equation of refraction is

$$\mu' \sin \varphi' = \mu \sin \varphi,$$

and from the figure the relations,

$$\cos \lambda = \sin \tau \cos (\pi - \zeta),$$

$$\sin \varphi = \frac{\mathrm{U} \sin \lambda}{\mathrm{R}},$$

$$\lambda' = \lambda + \varphi - \varphi',$$

$$\mathrm{U}' = \frac{\mathrm{R} \sin \varphi'}{\sin \lambda'} = \mathrm{U} \frac{\mu \sin \lambda}{\mu' \sin \lambda'},$$

$$\frac{\sin \tau' \sin (\pi' - \zeta)}{\sin \lambda'} = \frac{\sin \tau \sin (\pi - \zeta)}{\sin \lambda},$$

$$\sin \tau' \cos (\pi' - \zeta) = \cos \lambda'$$

are found.

Let the refracted ray meet the next surface whose centre is M_1 in P_1; and let the notation applied to the previous surface hold throughout for this surface with the addition of the suffix 1. The plane containing PQ and $\mathrm{P}_1\mathrm{Q}_1$ is parallel to the axis, and $\mathrm{Q}_1\mathrm{O}_1$ meets PQ in a point S in this plane. The intersection of this plane with the plane $\mathrm{MM}_1\mathrm{N}'$ is the line $\mathrm{N}'\mathrm{N}_1$, and this is parallel to the axis. Thus $\pi_1 = \pi'$, $\mathrm{QN}' = \mathrm{SN}_1$, and the distances of QN′ and SN_1 from the axis are equal, leading to

$$\mathrm{U}_1 \sin (\pi' - \zeta_1) = \mathrm{U}' \sin (\pi_1 - \zeta),$$

and

$$\mathrm{U}_1 \cos (\pi' - \zeta_1) = \mathrm{U}' \cos (\pi' - \zeta) - c \tan \tau',$$

where $c = \mathrm{MM}_1$. The four co-ordinates U, ζ, τ, π are selected to define the incident ray at one surface, and the above equations determine the corresponding quantities for the next surface, since the equation $\tau_1 = \tau'$ follows by definition.

These equations have the advantage of being in a form suitable for logarithmic computation, but the process is very tedious—nine equations have to be solved for each surface—and the method does not readily indicate what modifications should be made in the system when the ray does not emerge as is desired. An algebraic method is preferable precisely

because it is free from this objection, and the fact that it is not as well suited for logarithmic work is of no importance where a calculating machine is employed.

The algebraic formulæ to be used should, if possible, be comparable with those used for calculating paraxial rays. Let the system consist of n refracting surfaces, distinguished by the numbers 1, 2, 3, . . . λ . . . n, the radii of curvature being r_1, r_2, . . . r_λ, . . . r_n reckoned positive when the surfaces are convex to incident light passing from surface 1 to surface n. The refractive index of the medium between surfaces λ and $\lambda+1$ will be denoted by μ_λ, and the axial separation of these surfaces by t_λ. The power of the surface λ is $\kappa_\lambda=\frac{\mu_\lambda-\mu_{\lambda-1}}{r_\lambda}$, the sign being that commonly used by opticians. The power of the complete system and other constants are found by successive applications of the formulæ

$$\kappa_{1,\lambda}=\kappa_{1,\lambda-1}+\kappa_\lambda\frac{\partial\kappa_{1,\lambda}}{\partial\kappa_\lambda}, \qquad \kappa_{\lambda,n}=\kappa_{\lambda+1,n}+\kappa_\lambda\frac{\partial\kappa_{\lambda,n}}{\partial\kappa_\lambda},$$

$$\frac{\partial\kappa_{1,\lambda}}{\partial\kappa_\lambda}=\frac{\partial\kappa_{1,\lambda-1}}{\partial\kappa_{\lambda-1}}-\frac{t_{\lambda-1}}{\mu_{\lambda-1}}\kappa_{1,\lambda-1}, \qquad \frac{\partial\kappa_{\lambda,n}}{\partial\kappa_\lambda}=\frac{\partial\kappa_{\lambda+1,n}}{\partial\kappa_{\lambda+1}}-\frac{t_\lambda}{\mu_\lambda}\kappa_{\lambda+1,n},$$

where $\kappa_{1,\lambda}$ is the power of the system composed of the surfaces 1, 2, 3, . . . λ.

NOTE.—The advantage of using the second set of formulæ given above, instead of those which give $\frac{\partial\kappa_{1,n}}{\partial\kappa_1}$ and $\frac{\partial^2\kappa_{1,n}}{\partial\kappa_1\partial\kappa_n}$ directly is that in numerical calculations, which from their nature are approximations, the equality of the two values found for $\kappa_{1,n}$ is a better check on the accuracy of the result obtained than the satisfaction to the same number of significant figures of the identity

$$\frac{\partial\kappa_{1,n}}{\partial\kappa_1}\cdot\frac{\partial\kappa_{1,n}}{\partial\kappa_n}-\kappa_{1,n}\frac{\partial^2\kappa_{1,n}}{\partial\kappa_1\partial\kappa_n}=1.$$

If, now, a paraxial ray enters the system inclined at an angle θ_0 with the axis and meets the first surface at a distance S_1 from the axis, the emergent angle θ_n and the distance S_n at which the last surface is met, are given by the equations

$$\mu_n\theta_n=\mu_0\theta_0\frac{\partial\kappa_{1,n}}{\partial\kappa_1}-S_1\kappa_{1,n},$$

$$S_n=-\mu_0\theta_0\frac{\partial^2\kappa_{1,n}}{\partial\kappa_1\partial\kappa_n}+S_1\frac{\partial\kappa_{1,n}}{\partial\kappa_n};$$

or as an alternative to the latter,

$$\mu_0\theta_0=\mu_n\theta_n\frac{\partial\kappa_{1,n}}{\partial\kappa_n}+S_n\kappa_{1,n}.$$

Turning, now, to the general case, let a ray passing between surfaces λ and $\lambda+1$ have direction cosines L_λ, M_λ, N_λ referred to rectangular axes of which the axis of x coincides with the optical axis, and let d_λ be the length of that part of the ray which lies between these two surfaces, which must be symmetrical about the optical axis, but need not for the moment be assumed spherical. The direction cosines of the normal to the λth surface at the point of refraction will be denoted by l_λ, m_λ, n_λ, and the length of the normal intercepted between the optical axis and the surface by r_λ, so that r_λ is the transverse radius of curvature of the surface at the point of refraction.

The equations of refraction are

$$\frac{\mu_\lambda L_\lambda - \mu_{\lambda-1} L_{\lambda-1}}{l_\lambda} = \frac{\mu_\lambda M_\lambda - \mu_{\lambda-1} M_{\lambda-1}}{m_\lambda} = \frac{\mu_\lambda N_\lambda - \mu_{\lambda-1} N_{\lambda-1}}{n_\lambda}. \quad . \quad (1)$$

Denote each of these equal quantities by $r_\lambda K_\lambda$; K_λ is defined as the power of the surface for this particular ray. If D_λ is the deviation suffered by the ray the way in which K_λ varies with the obliquity of the incidence is indicated by the equation,

$$r_\lambda^2 K_\lambda^2 = \mu_\lambda^2 + \mu_{\lambda-1}^2 - 2\mu_\lambda \mu_{\lambda-1} \cos D_\lambda.$$

The equations giving d_λ are

$$d_\lambda = \frac{a_\lambda/\mu_\lambda + l_\lambda r_\lambda - l_{\lambda+1} r_{\lambda+1}}{L_\lambda} = \frac{m_\lambda r_\lambda - m_{\lambda+1} r_{\lambda+1}}{M_\lambda} = \frac{n_\lambda r_\lambda - n_{\lambda+1} r_{\lambda+1}}{N_\lambda}, \quad (2)$$

where a_λ/μ_λ is the intercept on the axis between normals to surfaces λ and $\lambda+1$ at the points where they are met by the ray. From equations (1) and (2) it is seen that if $K_{1,n}$, &c., are defined by the formulæ,

$$\left.\begin{aligned} K_{1,\lambda} &= K_{1,\lambda-1} + K_\lambda \frac{\partial K_{1,\lambda}}{\partial K_\lambda}, \\ \frac{\partial K_{1,\lambda}}{\partial K_\lambda} &= \frac{\partial K_{1,\lambda-1}}{\partial K_{\lambda-1}} - \frac{d_{\lambda-1}}{\mu_{\lambda-1}} K_{1,\lambda-1}, \\ &\&c., \end{aligned}\right\} \quad . \quad . \quad . \quad (3)$$

the exact equations for refraction through the system are

$$\left.\begin{aligned} \mu_n M_n &= \mu_0 M_0 \frac{\partial K_{1,n}}{\partial K_1} + m_1 r_1 K_{1,n}, \\ \mu_n N_n &= \mu_0 N_0 \frac{\partial K_{1,n}}{\partial K_1} + n_1 r_1 K_{1,n}, \\ m_n r_n &= \mu_0 M_0 \frac{\partial^2 K_{1,n}}{\partial K_1 \partial K_n} + m_1 r_1 \frac{\partial K_{1,n}}{\partial K_n}, \\ n_n r_n &= \mu_0 N_0 \frac{\partial^2 K_{1,n}}{\partial K_1 \partial K_n} + n_1 r_1 \frac{\partial K_{1,n}}{\partial K_n}, \end{aligned}\right\} \quad . \quad . \quad (4)$$

or, as alternative forms to the two latter equations,

$$\mu_0 M_0 = \mu_n M_n \frac{\partial K_{1,n}}{\partial K_n} - m_n r_n K_{1,n},$$

$$\mu_0 N_0 = \mu_n N_n \frac{\partial K_{1,n}}{\partial K_n} - n_n r_n K_{1,n}$$

may be used.

If, then, sufficiently simple formulæ can be obtained for the determination of the K's and d's, all rays can be traced through an axial system employing exactly the same formulæ as are used for paraxial rays. The required expressions evidently involve the shape of the section of the refracting surfaces by an axial plane. The most important case is that in which all the surfaces are spherical, so that $\frac{a_\lambda}{\mu_\lambda}$ is the distance between the centres of curvature of surfaces λ and $\lambda+1$. Let the lengths of the perpendiculars let fall on to the path of the ray in medium μ_λ from the centres of curvature of these two surfaces be $\frac{h_\lambda}{\mu_\lambda}$ and $\frac{h_{\lambda+1}}{\mu_\lambda}$ respectively. Then

$$\mu_\lambda d_\lambda = L_\lambda a_\lambda + \nu\, \mu_\lambda^2 r_\lambda^2 - h_\lambda^2 - \sqrt{\mu_\lambda^2 r_{\lambda+1}^2 - h_{\lambda+1}^2} \quad . \quad (5)$$

Also from the law of refraction the length of the perpendicular from the centre of curvature of surface λ to the incident ray is $\frac{h_\lambda}{\mu_{\lambda-1}}$, that is to say h_λ is unaltered by refraction.

It follows from the definition given of K_λ that

$$r_\lambda^2 K_\lambda = \sqrt{\mu_\lambda^2 r_\lambda^2 - h_\lambda^2} - \sqrt{\mu_{\lambda-1}^2 r_\lambda^2 - h_\lambda^2}. \quad . \quad . \quad . \quad (6)$$

Now let $l_{\lambda+1}$, $m_{\lambda+1}$, $n_{\lambda+1}$ be eliminated from equations (2), using the relation $l_{\lambda+1}^2 + m_{\lambda+1}^2 + n_{\lambda+1}^2 = 1$, and let d_λ be given its value from equation (5). The result is

$$h_{\lambda+1}^2 = h_\lambda^2 - 2a_\lambda L_\lambda \sqrt{\mu_\lambda^2 r_\lambda^2 - h_\lambda^2} + 2a_\lambda a_\lambda r_\lambda l_\lambda + a_\lambda^2 (1 - L_\lambda^2). \quad (7)$$

Equations (5), (6) and (7), together with

$$\mu_\lambda L_\lambda = \mu_{\lambda-1} L_{\lambda-1} + l_\lambda r_\lambda K_\lambda$$

and

$$l_\lambda r_\lambda = \frac{a_{\lambda-1}}{\mu_{\lambda-1}} + l_{\lambda-1} r_{\lambda-1} - L_{\lambda-1} d_{\lambda-1},$$

determine all the K's and d's of the system as was required. Since l_n and L_n have been obtained in the course of this preliminary work, the final result may be checked by verifying the equations

$$l_n^2+m_n^2+n_n^2=L_n^2+M_n^2+N_n^2=1$$

$$h_n^2=\mu_n^2 r_n^2\{1-(L_n l_n+M_n m_n+N_n n_n)^2\}$$

and

$$\mu_n r_n\{M_n n_n-N_n m_n\}=\mu_0 r_1\{M_0 n_1-N_0 m_1\}$$

$$=\frac{\mu_0\mu_n(M_0N_n-N_0M_n)}{K_{1,n}}.$$

When the performance of the system is unsatisfactory, it appears probable that a comparison of the K's and d's with the corresponding κ's and t's will suggest what modifications are necessary to amend the system in the direction desired.

It is of interest to notice some deductions from equations (4). Suppose x, y, z are the co-ordinates of a point on the incident ray and ξ, η, ζ those of a point on the refracted ray, the origins of both systems of co-ordinates being on the axis. If $\mu_0 P_0$ is the length of the incident ray between x, y, z and the first surface, and $\mu_n P_n$ the length of the emergent ray between ξ, η, ζ and the last surface,

$$\left.\begin{aligned} y&=-m_1 r_1-\mu_0 M_0 P_0 \\ z&=-n_1 r_1-\mu_0 N_0 P_0 \\ \eta&=-m_n r_n+\mu_n M_n P_n \\ \zeta&=-n_n r_n+\mu_n N_n P_n \end{aligned}\right\} \quad \ldots\ldots \quad (8)$$

or from equations (4)

$$K_{1,n}\eta=-\mu_0 M_0\left\{\left(P_0 K_{1,n}-\frac{\partial K_{1,n}}{\partial K_1}\right)\left(P_n K_{1,n}-\frac{\partial K_{1,n}}{\partial K_n}\right)-1\right\}$$

$$-K_{1,n} y\left(P_n K_{1,n}-\frac{\partial K_{1,n}}{\partial K_n}\right)$$

$$K_{1,n}\zeta=-\mu_0 N_0\left\{\left(P_0 K_{1,n}-\frac{\partial K_{1,n}}{\partial K_1}\right)\left(P_n K_{1,n}-\frac{\partial K_{1,n}}{\partial K_n}\right)-1\right\}$$

$$-K_{1,n} z\left(P_n K_{1,n}-\frac{\partial K_{1,n}}{\partial K_n}\right).$$

Now let ξ, η, ζ be the point on the refracted ray which lies in the same axial plane as x, y, z. Then $\frac{x}{y}-\frac{\xi}{\eta}=G$, say.

This requires

$$\left.\begin{aligned} P_0K_{1,n}&=\frac{\partial K_{1,n}}{\partial K_1}-\frac{1}{G}\\ P_nK_{1,n}&=\frac{\partial K_{1,n}}{\partial K_n}-G \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (9)$$

formulæ exactly corresponding to those for the determination of pairs of conjugate foci on the axis. If, then, conjugate points on a general ray are defined as the intersections of the incident and emergent portions of any ray with any plane through the axis of the system, all laws relating to conjugate foci on the axis will also hold of conjugate points on any ray; instead of transverse magnification, the ratio of the distances of the two points from the axis is to be taken; and instead of principal foci, principal points for the ray which are conjugate to the points at infinity on the ray.

It at once follows that the image of a plane can only be free from curvature for more than one magnification, if the rays before and after refraction are equally inclined to the optical axis. For assuming that for magnification G_0 the object and image are plane, when the object has been moved along the axis through a distance $\frac{1}{\kappa_{1,n}}\left(\frac{1}{G}-\frac{1}{G_0}\right)$, the object point on the ray considered will have been moved through a distance $\frac{1}{L_0\kappa_{1,n}}\left(\frac{1}{G}-\frac{1}{G_0}\right)$, and since the new magnification is to be G, this must be equal to $\frac{1}{K_{1,n}}\left(\frac{1}{G}-\frac{1}{G_0}\right)$, or $K_{1,n}=L_0\kappa_{1,n}$.

Similarly, the movement of the image plane is $\frac{1}{\kappa_{1,n}}(G-G_0)$ along the axis and $\frac{1}{L_n\kappa_{1,n}}(G-G_0)$ or $\frac{1}{K_{1,n}}(G-G_0)$ along the ray.

Thus $K_{1,n}=L_n\kappa_{1,n}$ must be satisfied if there is no curvature for the new magnification.

From the two values of $K_{1,n}/\kappa_{1,n}$ it is seen that L_n must be equal to L_0, the result stated.

If m_1, n_1, &c., are eliminated from equations (8) by means of

(4), and P_0 and P_n are given the values indicated in (9), the results are

$$y=\frac{1}{GK_{1,n}}(\mu_0 M_0-G\mu_n M_n),$$

$$z=\frac{1}{GK_{1,n}}(\mu_0 N_0-G\mu_n N_n),$$

$$\eta=\frac{1}{K_{1,}}(\mu_0 M_0-G\mu_n M_n),$$

$$\xi=\frac{1}{K_{1\,n}}(\mu_0 N_0-G\mu_n N_n),$$

that is to say, the co-ordinates of any pair of conjugate points satisfy the relations

$$K_{1,n}=\frac{\mu_0 M_0}{\eta}-\frac{\mu_n M_n}{y}=\frac{\mu_0 N_0}{\zeta}-\frac{\mu_n N_n}{z}. \quad . \quad . \quad . \quad . \quad (10)$$

Equations (10) may be taken as a basis for the definition of conjugate points, in which case the definition holds when the ray considered lies in an axial plane. For such rays the one conjugate point marks the intersection of the ray with the radial focal line formed by rays passing through the other point. This follows from simple considerations of continuity: assuming the existence of such a focal line its co-ordinates η, ζ are given by (10) for a ray lying very close to the axial plane but not in it; and the values of $K_{1,n}$, M_0, M_n, N_0, N_n may be made to differ by less than an arbitrarily assigned small quantity from those for the ray actually in the axial plane. The assumption of the existence of the focal line is readily justified, for from symmetry the path of the ray in the axial plane has either a maximum or a minimum value.

ABSTRACT.

Trigonometrical formulæ have been used for tracing rays not lying entirely in one plane through optical systems, as these can readily be arranged in a form suitable for logarithmic calculation. When a calculating machine is available such computations can be carried out more expeditiously by using algebraic formulæ given in the Paper; in form these exactly correspond with the expressions for paraxial rays, and a comparison of the numerical result appears likely to suggest what alterations should be made when a general ray does not behave as desired. It is pointed out that if the two points in which a general ray meets an axial plane are defined as conjugate points, all pairs of conjugate points on a ray are connected by the same relations as hold for object and image points for paraxial rays,

and the theory for paraxial rays can be extended to rays in general by placing a suitable interpretation on magnification, &c. The definition of conjugate points can be extended to include rays lying in axial planes, in which case the one point marks the intersection of the ray with the radial focal line formed by rays passing through its conjugate.

DISCUSSION.

Mr. S. D. CHALMERS was particularly interested in the method of tracing rays through a system. The relationships between the various quantities involved was of considerable interest. The author's methods of treatment seem to present many possibilities in addition to the cases actually quoted. By extending them he may simplify very considerably many important computations. Some time ago he had used somewhat similar methods to express the image position of an object in terms of the separations of the surfaces and the height of the intercepts of a standard ray on the different surfaces.

Mr. T. SMITH, in reply, agreed that if one had time to go into all the cases which might arise many interesting and useful results would follow. It was easy from the formulæ he had quoted to obtain an idea of what other general relations are likely to exist in an optical system, and to extend the meaning to be attributed to various conditions, such, for example, as the sine condition of Abbé. He hoped to treat more fully of these matters later.

XXXVIII. *On an Investigation of the Accuracy of the Lens and Drop Method of Measuring Refractive Index.* *By* H. Redmayne Nettleton, *B.Sc., Assistant Lecturer in Physics at Birkbeck College.*

Received May 10, 1915.

1. *Introduction.*

If a few drops of liquid be introduced between a plane mirror and a thin convex lens we have in effect a combination of the latter with a plano-concave lens of the liquid. From a knowledge of the curvature of the lower face of the convex lens together with its focal length and that of the combination, the refractive index of the liquid used can be calculated.

This method is described by T. H. Blakesley ("Proc." Opt. Con., 1905, p. 189). It is also given in "Nature," Vol. LXXXVII., Oct. 26, 1911, and in Dr. Clay's "Treatise on Practical Light," and has for long afforded a laboratory exercise, the focal distances being almost invariably measured by the pin and parallax method probably on account of the fact that the lens and mirror most conveniently lie horizontally. Thus, as usually practised the method is very rough, and no notice of it is taken by F. E. Wright in his discussion on the accuracy of the many drop methods in use (Washington Acad. Sci. "Journal" IV., pp. 269-279, June, 1914).

The object of the present communication is to show that by the choice of lenses of special curvature, and with the aid of totally reflecting prisms the method can be adapted to the optical bench and monochromatic light to give results reliable at least to the third place of decimals—an accuracy nearly always sufficient for the petrographer or the physical chemist in his determinations of molecular refractive powers. For substances having refractive index lying between 1·33 and 1·4 —which will include most aqueous solutions of salts and mixtures of alcohol and water—an accuracy in the value of μ is obtainable of about 0·0002. Thus, this method usually so roughly practised can, by the careful design of the lenses and by taking into account their thickness, be transformed with little loss of simplicity into one of the most accurate and rapid laboratory experiments on refractive index. Strictly, the apparatus here described is one for comparing refractive

indices, for it is necessary to find the constant of the instrument with a liquid of known refractive index. There is, however, no difficulty here, as the refractive index of distilled water is known at ordinary temperatures with an accuracy of practically the fifth decimal place.

2. *Simple Theory of the Method Neglecting Thickness of the Lens.*

The simple theory is well known, but is reproduced very briefly here for convenience in consequence of what follows in section 3 of this Paper.

Let f=focal length of the thin convex lens,

φ=focal length of the plano-concave lens formed of the liquid of refractive index μ_x,

F_x=focal length of the combination,

r=radius of curvature of the lower face of the convex lens, which is also that of the concave lens.

Then, by the law of combination of lenses we have numerically :—

$$1/\varphi = 1/f - 1/F_x = (\mu_x - 1)/r,$$

or,
$$\mu_x - 1 = \frac{r(F_x - f)}{F_x \cdot f}.$$

Similarly, for water or other standard substance,

$$\mu_s - 1 = \frac{r(F_s - f)}{F_s \cdot f}.$$

[illegible]

whence
$$\frac{\mu_x - 1}{\mu_s - 1} = \frac{(F_x - f)F_s}{(F_s - f)F_x}.$$

Let $F_x - f$, which is measurable directly, $= X$.

Let $F_s - f$, likewise measurable directly, $= S$.

Then,
$$\mu_x = 1 + (\mu_s - 1)\frac{(S+f)X}{(X+f)S}.$$

On inspecting this formula it will at once be seen that the accuracy of the method depends on the measurement of differences of focal lengths. A small error made in determining f the focal length of the convex lens—which can be found once and for all—is relatively unimportant unless S and X are small or differ greatly. By using a standard liquid,

such as water, we do away with the necessity of measuring a radius of curvature and virtually find the constant K in the formula which the method assumes, viz :—

$$x=\mu_x-1=\mathrm{K}\left(\frac{\mathrm{F}_x-f}{\mathrm{F}_x}\right).$$

It is of importance to find out whether the effect of the thickness of the lens is to merely modify the constant K, which for a lens infinitely thin is equal to r/f.

3. *Effect of the Thickness of the Lens.*

The effect of the thickness of the lens can be found by considering the full refraction of the parallel rays returning from normal reflection by the plane mirror.

Applying the formula :—

$$\frac{\mu}{\mathrm{V}}-\frac{1}{\mathrm{U}}=\frac{\mu-1}{r},$$

which holds strictly for determining "limiting foci" at spherical surface, let μ_x equal the refractive index of the liquid

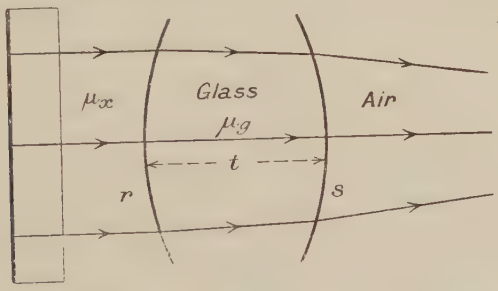


Fig. 1.

placed between the mirror and face r of the lens, and let μ_g equal the refractive index of the glass, all refractive indices being with reference to that of air taken as unity. Then, for the first refraction we have :—

$$\frac{\mu_g/\mu_x}{\mathrm{V}'}-\frac{1}{\mathrm{U}}=\frac{(\mu_g/\mu_x)-1}{r},$$

or, since for this refraction, $\mathrm{U}=\infty$,

$$\mathrm{V}'=\frac{r\,.\,\mu_g}{\mu_g-\mu_x}.$$

If t be the thickness of the lens and s the radius of curvature

of the face remote from the mirror, we have for the second refraction from glass to air :—

$$U = V' + t,$$

$V = F$ (the focussing distance from the face s),

and, hence :—

$$\frac{1/\mu_g}{F} - \frac{1}{t + \frac{r}{1-\mu_x/\mu_g}} = \frac{(1/\mu_g)-1}{s}, \quad . \quad . \quad . \quad . \quad (1)$$

whereas, if air replace the liquid, x

$$\frac{1/\mu_g}{f} - \frac{1}{t + \frac{r}{1-1/\mu_g}} = \frac{(1/\mu_g)-1}{s}, \quad . \quad . \quad . \quad . \quad (1a)$$

hence,

$$\frac{F-f}{Ff} = \frac{r(\mu_x - 1)}{(t + r - t/\mu_g)(t + r - t \cdot \mu_x/\mu_g)}. \quad . \quad . \quad (2)$$

Transposing and writing x for $\mu_x - 1$, we obtain :—

$$x = K'\left(\frac{F-f}{F}\right)\left(1 - \frac{t}{r+t} \cdot \frac{\mu_x}{\mu_g}\right). \quad . \quad . \quad . \quad (3)$$

where $$K' = \frac{(r+t)(r+t-t/\mu_g)}{fr} = \text{a constant.}$$

Applying formula (3) first to the substance of refractive index μ_x and, secondly, to the standard substance of refractive index μ_s, as in section 2, we obtain algebraically :—

$$x - (\mu_s - 1)\left[\frac{F_x - f}{F_s - f} \cdot \frac{F}{F_x}\right]\left(1 - \frac{\mu_x - \mu_s}{(r - t/\mu_g) - \mu_s}\right).$$

Expressing the measurable distances of withdrawal $F_x - f$, $F_s - f$, by X and S respectively as before, and remembering that r is negative, we may write arithmetically, that is taking all symbols as positive :—

$$x = (\mu_s - 1)\left[\frac{S - f}{X + f} \cdot \frac{X}{S}\right]\left(1 - \frac{\mu_x - \mu_s}{T}\right). \quad . \quad . \quad . \quad . \quad (4)$$

where

$$T = \frac{(r-t)\mu_g}{t} + \mu_s. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The use of a standard liquid thus only in part corrects for the thickness of the lens. The correcting factor, however, is small, and the denominator T is found without trouble, once and for all, in the following manner :—

The thickness of the lens t is found with a screw-gauge.

The radius of curvature of the drop face r is found without any additional measurement—with an accuracy far greater than is needed—from equation (2) above which may be written thus :—

$$\frac{x\mathrm{F}f}{\mathrm{F}-f} = \{r+t(1-1/\mu_g)\}\left\{1+\frac{t}{r}(1-\mu_x/\mu_g)\right\}, \quad . \quad . \quad . \quad . \quad (6)$$

$$= r+t(1-1/\mu_g)+t(1-\mu_x/\mu_g) \text{ very approximately.}$$

Thus, algebraically, $r=\dfrac{x\mathrm{F}f}{\mathrm{F}-f}-t(2-1/\mu_g-\mu_x/\mu_g)$ very approximately.

Whence, taking F as the focussing distance when using water, putting $x=\frac{1}{3}$ and $\mu_g=1{\cdot}5$, we have, taking all symbols with the positive sign :—

$$r=\frac{1}{3}\frac{\mathrm{F}f}{\mathrm{F}-f}+\frac{4}{9}t. \quad . \quad . \quad . \quad . \quad . \quad (7)$$

The accuracy obtainable in this method of measuring curvature is limited in practice by the accuracy of the measurements of f and $\mathrm{F}-f$, since in thin lenses the approximations introduce no appreciable errors. If the lens were very thick, and μ_g differed greatly from 1·5 formula (6) could always be applied, μ_g being found as described below. If $\mathrm{F}-f$ should be too small to measure accurately benzene should be used instead of water, whence, since μ_x/μ_g will be very nearly unity, we have with an even closer theoretical accuracy :—

$$r=\frac{x\mathrm{F}f}{\mathrm{F}-f}+\frac{1}{3}t. \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The only other term needed to find T is the refractive index of the glass of the lens which in a small correcting factor might be taken as 1·5. It is, however, easy to find μ_g more accurately once and for all in the following simple way. Reversing the lens, introduce between the mirror and the face of curvature s, usually uppermost, a few drops of benzene or oil of cedar wood, and find F', the focussing distance of the combination measured from the face r. The refractive index of the liquid used is not

required, but it must be near that of the glass of the lens. We have then, interchanging r and s in formula (1) :—

$$\frac{1/\mu_g}{F'}-\frac{1}{t+\dfrac{s}{1-\mu_x/\mu_g}}=\frac{1/\mu_g-1}{r}, \quad . \quad . \quad . \quad . \quad (1B)$$

or, as s is always large in suitable lenses, and μ_x/μ_g is very nearly unity :—

$$\frac{1}{F'}=\frac{1-\mu_g}{r} \text{ very approximately,}$$

whence, arithmetically, $\mu_g=1+\dfrac{r}{F'}$, ... (9)

and this value of μ_g could be used in equation (6) in finding r with precision in the case of a very thick lens.

As the refractive index of the glass of a lens is sometimes required very accurately a typical example of how this may be done is given in section 6, subsection (*d*) below :—

The approximate value of $\mu_x=1+x$, found by neglecting the correcting factor altogether, of course, serves for the numerator $\mu_x-\mu_s$ of this fraction. Typical experiments are given in section 6 below.

4. *Description of Suitable Apparatus.*

A form of suitable apparatus will be readily understood from the diagram Fig. 2. It consists essentially of two parts—an arrangement for supporting the right-angled prism P and lens and mirror M, and an optical bench and object-screen sliding piece, S. For the former purpose a good heavy adjustable "Young's Modulus table," with levelling screws, was found suitable, a small piece of metal being cut away over which the prism P stood fixed as shown. The table top T could be slightly rotated. The double platform piece D was clamped at L to the vertical table pillar, and afforded a support for the block of wood W carrying the lens and mirror. This mirror, about an inch square, could be adjusted to the horizontal by the screws s, and was cemented with red lead to the block W, which should always be replaced on the table, touching the same margin line.

The optical bench, 120 cms. long, is a good one, such as used for the bi-prism experiment, and one of its disused but well

made carrier pieces S, fitted with a vernier reading to 0·1 mm., was easily adapted for receiving an object-screen. A postcard with a small circular hole punched out of it answered this

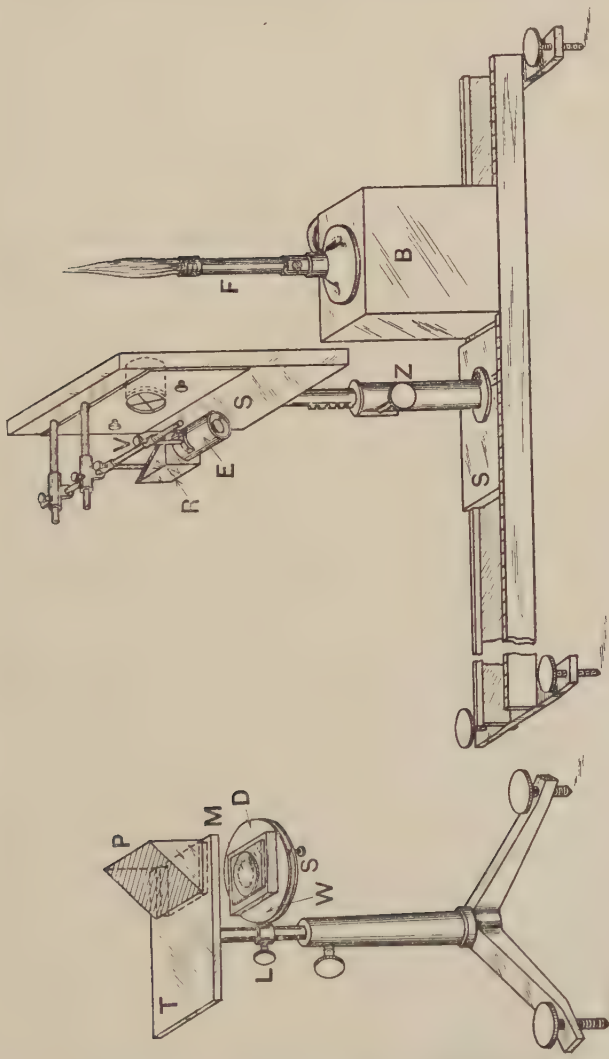


Fig. 2.

purpose well, when furnished with threads in the form of a square with diagonals, the diagonal cross-wires being very fine. This object was illuminated from behind by the sodium flame

F standing on the block B, which could easily be moved along the optical bench. When in the focal plane of the lens and drop combination, the reflected image of the cross-wires could be received either on the far side of the object on the screen itself, or else on the near side by the small right-angled prism R, which turned it into the focal plane of the negative microscopic eye-piece E, supported as shown. This latter device very effectively overcomes the difficulty of obtaining sufficient illumination, especially at relatively great focal distances, where without it the room must be quite dark, and the flame worked up to its best. On the other hand, using the small prism and eye piece the image is perfectly clear even when all the lights are on in the room, and the sodium flame is feeble. Moreover the definition of the image of the fine cross-wires in the microscope eye-piece is such that the whole carrier-piece S can be set to a consistency of a millimetre at a focal distance of quite a metre. The eye-piece must be set by the observer once and for all in the manner described below.

The adjustment of the apparatus prior to use is simple. The table T is first levelled to the horizontal, and then also the platform D and plane of the mirror by means of the screws *s*. The optical bench is likewise levelled, and the height of the object-screen brought to the position which, at the right focal range, gives image and object side by side in a horizontal line when a lens lies on the mirror. By very slightly rotating the table platform T the image can then be turned into the prism R and eye-piece. The position of the eye-piece must then be adjusted on its supporting arrangement, such as that shown, so that when the image is in the sharpest focus on the object-screen, as seen under the best conditions of illumination, it is likewise in sharpest focus when viewed through the eye piece. If the sodium flame is not very good an incandescent burner should be used for this purpose, a signal green glass being placed between burner and object-screen. If the prism R is properly placed the agreement, once obtained, will be found practically perfect everywhere, except for positions of very short focal range, where there is, of course, no necessity to use the eye-piece arrangement at all. For such testing purposes it is convenient to have at hand for placing on the mirror two common spectacle lenses of focal lengths about 50 cm. and 100 cm. respectively. These adjustments can be made in a few minutes, and most of them are permanent: when the eye-piece is set it should be firmly clamped at V, and

all screws tightened. Care should be taken not to move the prism P, and no means of bringing the image into its proper position on the object-screen or into the microscope eye-piece should be attempted other than a slight turn of the table T, the sliding of the entire screen piece, and if need be the slightest vertical movement of the object-screen by means of the screw Z.

In performing an experiment the lens selected is placed with the face of radius r symmetrically on the mirror, and the block W placed in position. The "zero reading" on the optical bench is then obtained when the image of the cross-wires is formed on the screen alongside the object. Two or three drops of distilled water are then placed on the mirror, and the lens replaced. A reading is now obtained through the eye-piece, and on subtracting the zero reading we obtain S, the distance of withdrawal for water. Similarly the distance of withdrawal X for the liquid under investigation is obtained. In both cases the temperature should be taken as recorded by a thermometer, subsequently placed with its bulb between the mirror and peripheral portion of the lens. Care should be taken not to warm the lens with the hand, or, at all events, time given for the temperature to recover and the focussing length to attain a steady value. Owing to the smallness of the exposed surface of the liquid evaporation is exceedingly slow; none the less it is desirable to devise a water circulation to maintain steadier conditions. The focussing length f is, of course, obtained in the ordinary manner on any optical bench employing sodium light by measuring the distance from the nearest face s of the lens to the coincident image and object. It is easily obtained correct to the nearest 0·5 mm., and is a constant, being practically independent of temperature changes.

It should be noted that the effect of the prism P is to lengthen all focussing distances by the same amount, not affecting, therefore, differences of focal lengths. Clearly actual focal lengths can be found by the periscopic arrangement by employing as a "distance-piece" any lens of known focal length and thickness.

5. *Selection of Mirror, Lenses and Standard Substances.*

I. It is absolutely essential that the small mirror, an inch square, be of good parallel plane glass—otherwise the image will show apparent astigmatism, vertical and horizontal lines

having different focal distances. There is no need, however, to resort to optically worked glass, as it is easy to select specimens of patent plane mirror glass which are quite satisfactory. Perhaps a trace of "astigmatism" is if anything helpful, enabling a sharp setting of the cross-wires for equality of definition.

11. The whole accuracy of the method is intimately connected with the careful prescription of the lenses. They should be thin, of aperture about 1 in. to 1·5 in., and made of glass of refractive index of about 1·5. In making the selection for any purpose free use should be made of the approximate formula—

$$F = fr/(r-xf),$$

and it should at the same time be remembered that while the "range of good focus" is proportional to F, the sensitiveness $dF/dx = F^2/r$, and absolute measurements of short focussing distances are, therefore, to be avoided.

The following set of four lenses to which special attention has been given illustrate well the possibilities and limitations of the method. The approximate dimensions to which they were ordered are given in centimetres :—

A. A bi-convex lens of unequal radii of curvature.
$f=20$; $r=12$; $s=60$ (approx.).

B. A bi-convex lens of unequal radii of curvature.
$f=30$; $r=20$; $s=60$ (approx.).

C. A concavo-convex lens or a converging meniscus.
$f=17$; $r=7{\cdot}5$; $s=64$ (approx.).

D. A bi-convex lens of unequal radii of curvature.
$f=20$; $r=15$; $s=30$ (approx.).

Experiments are given in section 6 below illustrating the respective uses of these lenses. It will be found that lenses A and B are applicable over practically the same range and thus serve well for comparative tests. The accuracy obtainable in measuring, say, a refractive index of 1·5 with lens A is dependent on the accuracy of measurement of the distance of withdrawal, X = 20·96 cm. for water. An error of 1 mm. in this measurement would affect the value of $\mu - 1$ for the unknown liquid by 1 part in 400. Owing, however, to the shortness of both focal distances the measurement can probably be made

correct to 0·05 cm. An error of 0·05 cm. in the measurement of f the focal length of the lens would affect the result by 1 part in 1,400. The change of focal length due to substituting benzene for water is over 50 cm.

Corresponding errors made with lens B would have less effect on the result; the correcting factor for thickness is smaller; and an error in the setting for the zero reading is of less importance. A change of focal length of some 50 cm. again results on replacing water by benzene.

Using the converging meniscus C the method is perhaps seen at its best, refractive indices up to 1·39 being certainly obtained correct to 0·0002. On substituting for water a strong salt solution of refractive index 1·3807, the focal length rose from 68 to 118 cm. Errors in measuring f or observing the zero are of minimum importance.

Lens D is specially for high refractive indices, and benzene should be used as the standard substance. For comparative purposes it is very accurate, but the absolute values are dependent on the data given for benzene below.

For ordinary laboratory use lenses C and either A or B will usually be found sufficient.

III. (*a*) Water is used as the fundamental standard substance. Its temperature coefficient is abnormally small, and its refractive index is known to the fifth decimal place. The values given below are based on those of Jamin, Lorenz, Walter and others, as given in the Physikalisch-Chemische Tabellen of Landolt, Börnstein and Roth. $\mu-1$ may thus often be taken $=1/3$ with sufficient accuracy.

Temperature.	Refractive index air to water.	Temperature coefficient.
20°C.	1·33300	0·00008
15°C.	1·33339	0·00007
10°C.	1·33369	0·00006

(*b*) Benzene will be found useful as a secondary standard, especially in determinations of high refractive indices. It can be crystallised and easily obtained pure, and its purity can be readily tested. Moreover, the presence of traces of impurity has no marked effect on the refractive index. A sample of "pure crystalline benzene" was found on a refractometer to differ in refractive index from Kahlbaum's pure benzene for molecular weights by not more than 0·0003 under constant temperature conditions. Benzene has the disad-

vantage, however, of a high temperature coefficient compared with that of water: when it is used, therefore, temperature must be carefully observed. Based on Weegmann, Knops and Landolt and Jahn we may take $\mu_{15} = 1{\cdot}5039$ and $\mu_{20} = 1{\cdot}5013$ with 0·00065 as the temperature coefficient. The nearness of the refractive index to that of glass is useful in determining the refractive index of the lens as explained in section 3 equation (9). For this latter purpose oil of cedar wood answers equally well.

6. *Typical Experiments on the Constants of Lenses and on Determinations of Refractive Indices.*

(*a*) Taking lenses A and B the following readings were obtained :—

TABLE I.—*Readings with Lenses A and B.*

Nature of observation.	A	B
Thickness of lens in centimetres	0·318	0·321
Focussing length correct to 0·05cm. (from nearest face *s*)	18·70	29·20
Zero on Optical Bench	4·95	15·45
Reading with oil of cedar between mirror and lens face *s*.	8·59	25·25
Readings with liquids between mirror and face *r* :—		
(*a*) Water, temperature 11·7°C.	25·91	42·52
(*b*) Chloroform, temperature 12·0°C.	51·25	69·45
(*c*) Benzene, temperatures 12·1°C.. 12·2°C.	78·90	93·50

Thus, with lens A we obtain—

Distance of withdrawal S for water $= 20{\cdot}96 : S - f = 39{\cdot}66.$

Distance of withdrawal X for benzene $= 73{\cdot}95 : X - f = 92{\cdot}65.$

$$\mu - 1 \text{ for water at } 12°\text{C.} = 0{\cdot}33357,$$

whence regarding the lens as infinitely thin, we obtain for benzene

$$\mu - 1 = 0{\cdot}33357 \times \frac{39{\cdot}66}{20{\cdot}96} \times \frac{73{\cdot}95}{92{\cdot}65},$$

$$= 0{\cdot}503_{75}, \text{ which is the very approximate value.}$$

The denominator T of the correcting factor for thickness is now found, and will serve for always.

By formula (7) we have for the radius of curvature of the drop face *r* :—

$$r = \tfrac{1}{3} \times 18{\cdot}70 \times \frac{39{\cdot}66}{20{\cdot}96} + \frac{4}{9} \times 0{\cdot}318,$$

$$= 11{\cdot}9_4 \text{ cm.}$$

And by formula (9) for the refractive index of the glass of the lens—

$$\mu_g = 1 + \frac{11{\cdot}94}{18{\cdot}70 + 8{\cdot}59 - 4{\cdot}95} = 1{\cdot}53,$$

and, hence, from (5)

$$T = \frac{11{\cdot}94 - 0{\cdot}32}{0{\cdot}318} \times 1{\cdot}53 + 1{\cdot}33,$$

$$= 57{\cdot}3.$$

Thus, we obtain for the corrected value for benzene

$$\mu - 1 = 0{\cdot}50378\left(1 + \frac{0{\cdot}5038 - 0{\cdot}3336}{57}\right),$$

$$= 0{\cdot}5038 + 0{\cdot}0015,$$

whence $\mu = 1{\cdot}505_3$ at 12·1°C.

Similarly, for chloroform,

$$\mu = 1 + 0{\cdot}4496\left(1 + \frac{0{\cdot}4496 - 0{\cdot}3336}{57}\right),$$

$$= 1{\cdot}450_5 \text{ at } 12{\cdot}0°\text{C}.$$

An Abbé refractometer with water circulation gave at 12·1°C.

for water $\mu = 1{\cdot}3334$,

for chloroform $\mu = 1{\cdot}4499$,

for benzene $\mu = 1{\cdot}5050$.

The readings for lens B give T = 96·6, which could be taken as T = 100, with sufficient accuracy as the correction for benzene is only 0·0009. The values obtained are :—

for chloroform $\mu = 1{\cdot}450_6$ at 12·0°C.

for benzene $\mu = 1{\cdot}505_6$ at 12·2°C.

(*b*) The converging meniscus C had focal length from near face = 16·80 cm. and thickness = 0·286 cm. Experiment showed $r = 7{\cdot}57$; $a\mu_g = 1{\cdot}53_5$; and T = 40·4 = 40 approx. It is very sensitive to small changes in refractive index, and all

necessary measurements can be made with a high accuracy. A curve of the refractive index and density of salt solutions was, therefore, attempted, at the same time a comparison with an Abbé refractometer being made. The temperature of the research laboratory was very constant, and the water circulating through the Abbé prisms was easily adjusted to this temperature. A specific gravity bottle containing the solution having been weighed, drops of liquid were placed on each refractometer. The comparison is tabulated below:—

TABLE II.—*Showing Readings with Converging Meniscus C.*

Specific volume of salt sol.		Refractive index by lens and drop method.				Refractive index by Abbé refractometer.	
Weight of contents of full sp. gr. bottle.	Temperature.	Reading on optical bench.	Temperature.	$\mu-1$ uncorrected.	μ corrected.	μ	Temperature.
70·705	12·6	104·5	11·9	0·38016	1·3806	1·3809	12·0
69·416	12·8	95·92	11·8	0·37524	1·3756	1·3759	11·8
67·871	12·5	87·60	12·0	0·36908	1·3700	1·3702	12·0
66·816	13·0	82·16	11·9	0·36552	1·3658	1·3660	12·0
66·161	13·3	79·30	11·9	0·36314	1·3634	1·3636	12·0
64·549	12·6	72·37	12·1	0·35672	1·3569	1·3571	12·0
63·607	13·9	68·90	12·1	0·35310	1·3533	1·3533	12·0
62·791	12·5	65·95	11·7	0·34977	1·3499	1·3500	12·0
61·819	13·0	62·60	12·0	0·34569	1·3458	1·3459	12·0
60·955	12·9	59·81	12·6	0·34200	1·3421	1·3419	12·0
60·253	13·0	57·82	12·3	0·33919	1·3392	1·3394	12·0
59·350	13·0	55·38	12·3	0·33553	1·3355	1·3354	12·0
*58·868	12·8	54·12	12·5	...	†1·3335$_4$	1·3334	12·0

* Water. † Assumed value.

The zero on the optical bench was at 2·95, and the reading with oil of cedar and the lens reversed was 0·30. Meniscus C can be used for concentrated common salt solutions which are at about its maximum limit. A second meniscus, E, with $f=18{\cdot}45$ and S the distance of withdrawal for water $=42{\cdot}75$ was accurate and serviceable for solutions up to $\mu=1{\cdot}41$.

(*c*) Lens D, suitable for high refractive indices gave the following data: Thickness $=0{\cdot}321$ cm.; $f=18{\cdot}80$ cm.; zero on bench $=5{\cdot}05$; reading with oil of cedar wood and lens reversed $=14{\cdot}00$; reading with water $=18{\cdot}90$ at 12·8°C.; reading with benzene $=38{\cdot}58$ at 12·6°C.; reading with carbon bi-sulphide (redistilled)$=81{\cdot}1$ at 12·0°C.

Owing to the smallness of the distance of withdrawal for water, benzene should be used as the standard, μ_s being taken

as $1{\cdot}506_2$. Formula (8) gives $r=14{\cdot}96$, and T will be found $=70$ approximately. Hence, for carbon bi-sulphide :—

$$\mu-1=0{\cdot}5062\times\frac{52{\cdot}33}{33{\cdot}53}\times\frac{76{\cdot}05}{94{\cdot}85}\text{ (approx.)},$$

$$=0{\cdot}633_4,$$

$$\mu-1\text{ (corrected)}=0{\cdot}6334\left(1+\frac{0{\cdot}6334-0{\cdot}5062}{70}\right),$$

$$=0{\cdot}6334+0{\cdot}0012,$$

or $$\mu=1{\cdot}634_6.$$

An Abbé refractometer gave for benzene 1·5048 at 12·7°C., and for carbon bi-sulphide 1·6336 at 12·5°C.

(*d*) It has already been shown how by the use of a liquid of refractive index near that of glass, the refractive index of the glass itself may be easily found approximately. If the trouble be taken to find by the method of this Paper the refractive index of the liquid used, the refractive index of the glass may be found to an accuracy similar to that which holds for r and f, and hence s may be deduced.

To illustrate this the benzene found by lens B, *see* sub-section (*a*) above, to have at 12·2°C. refractive index $1{\cdot}505_6$, was found when placed between the face s and the mirror (lens reversed) to give at 12·0°C. the reading 25·15 on the optical bench. Whence $F'=29{\cdot}20+25{\cdot}15-15{\cdot}45=38{\cdot}90$.

Thus, $\mu_g=1+r/F'=1+20{\cdot}39/38{\cdot}9=1{\cdot}52_{42}$ approx.

From equation (1a) and by other means we find $s=60$ approx. But from equation (1b) we have with great accuracy for $a\mu_g$

$$\mu_g=1+\frac{r}{F'}-\frac{r}{r+s}\left(1+\frac{r}{F'}-\mu_x\right),$$

all symbols being positive. Hence,

$$\mu_g=1{\cdot}5242-\frac{20{\cdot}39}{80{\cdot}39}\,(1{\cdot}5242-1{\cdot}5056),$$

$$=1{\cdot}5242-0{\cdot}0047,$$

$$=1{\cdot}520.$$

From (1) or (1a) we now obtain :—

$$s=60{\cdot}4\text{ cm.}$$

(*e*) An instance of the special usefulness of the lens and drop method is the following. The author determined for

his colleague, Dr. G. Martin, the refractive indices of a number of newly-prepared alcohol derivatives of silicon hexachloride (Martin, Chem. Soc. "Trans.," Vol. CV., 1914). These compounds fuming in the air leave a deposit of white solid on glass which is only removed by warming with caustic soda. The lens can easily be warmed in hot alkali, and the ring which protects the rest of the liquid, dissolved away, but similar treatment in a refractometer would be impossible.

7. *Summary and Conclusion.*

The lens and drop method of comparing refractive indices is a very accurate one when due regard is paid to the choice of lenses, and allowance made for the thickness. It is easily adapted to the optical bench and monochromatic light, and measurements may be made very quickly. The method is perhaps seen at its best when employing a suitable converging meniscus to measure refractive indices between 1·3 and 1·4. The method of measuring short radii of curvature in terms of the well-known refractive index of water is also frequently very useful and accurate just when other means are difficult or inaccurate. The lens and drop method may also be of service in finding the refractive index of the material of the lens. The apparatus here described is very suitable as providing a laboratory experiment for advanced students in view of the accuracy of the results obtained by the inexpensive adaptation of standard apparatus.

The author must express his special thanks to Dr. R. S. Willows, M.A., of the Sir John Cass Technical Institute, for the loan of a refractometer for comparative purposes, and to Dr. A. Griffiths, head of the Physics Department, Birkbeck College, for putting at his disposal all the other apparatus needed. He would also record his indebtedness to Mr. S. D. Chalmers, M.A., for suggestions made since the communication of this Paper, especially with reference to the notation adopted.

XXXIX. *Conduction of Electricity Through Metals.* *By* Sir J. J. Thomson, *O.M., F.R.S.*

Received June 25, 1915.

The investigations of Kamerlingh Onnes on the resistance of metals at the temperature of liquid helium have led to results which are of vital importance in the theory of metallic conduction ; they have shown, for example, that some metals can exist in a state where their specific resistance is less than one hundred thousand millionth part of that at 0°C. The transition from the state in which the resistance is diminishing normally with the temperature to the one where they possess this super-conductivity takes place abruptly at a definite temperature, and the difference in the electrical properties of the metal above and below this temperature are as well marked as the difference in elastic properties when a solid melts or in the magnetic ones when a piece of iron passes through the temperature of recalescence. One of the most remarkable effects discovered by Kamerlingh Onnes is that when a current, was started in a small ring of lead at a temperature of about 4 deg. absolute, by bringing a magnet close to it ; the current, instead of dying away as it would have done at 0°C. as soon as the magnet was stopped, went on with practically undiminished intensity, its rate of decay being so slow that Kamerlingh Onnes estimated that it would take four days to fall to half its initial value. This power of transmitting a current for long periods when no external E.M.F. acts on the metal is one that has to be accounted for by any theory of metallic conduction ; any such theory must indicate that in certain metals a change of electrical state takes place at a definite temperature, that above this temperature the current dies away almost instantaneously after the E.M.F. is removed, while below it the current may persist for days without undergoing any considerable diminution. It seems to me that this is another and fatal objection to the theory that metallic conduction is due to the presence in the metal of free electrons which drift under the electric force, for no permissible increase in the number of free electrons or in the mean free path would explain the difference between the ordinary and super-conducting state. In the case of the lead ring the maximum free path (equal to the longest chord that can be drawn in the ring) cannot be more than a few millimetres.

It is the object of this Paper to show that the effects discovered by Kamerlingh Onnes are in accordance with the theory of metallic conduction which I gave in "The Corpuscular Theory of Matter," page 85, and which, with the substitution of an electron for a charged atom is substantially the same as that given in my "Applications of Dynamics to Physics and Chemistry," 1888.

On this theory the atoms of some substances, including the metals, contain electrical doublets, *i.e.*, pairs of equal and opposite electrical charges at a small distance apart. In the normal state of a body the axes of the large number of doublets occurring in even a small volume are uniformly distributed in all directions; when, however, an electrical force acts on the body, the axes of the doublets tend to point in the direction of the force, and the moments of the doublets have a finite resultant in this direction. If the axes of the doublets were quite free to set in any direction, the smallest electrical force would be able to pull the axes of all the doublets into line and thus produce the maximum polarisation. There are, however, several influences at work which limit the number of doublets which point in the direction of the electric force.

In the case of gases, for example, there are collisions between the various molecules which tend to knock the axes of the doublets out of line as fast as they are brought into it by the electric force. Langevin has calculated from the principles of the "Kinetic Theory of Gases" the magnitude of this effect, and has shown that if M is the moment of each doublet, N the number of doublets in unit volume, I the resultant of these moments parallel to x and X the force on a doublet in this direction,

$$I = NM\left\{\frac{\varepsilon^{x}+\varepsilon^{-x}}{\varepsilon^{x}-\varepsilon^{-x}}-\frac{1}{x}\right\},$$

where $$x = MX/R\theta.$$

θ being the absolute temperature and $R\theta$ the mean kinetic energy of a molecule at this temperature; when x is very small $I=\frac{1}{3}NMx$, when it is very large $I=NM$.

In the case of solids and liquids, though there may not be collisions between the molecules, the rotation of the molecules endows them with a quasi rigidity, making each molecule behave very much as if its axis of rotation were acted on by a restoring couple proportional to the angle through which the axis is displaced and proportional also to the kinetic energy possessed

by the body in virtue of its rotation, it behaves, in fact, very much like a spring whose stiffness is proportional to its kinetic energy. The value of I will be a function of the ratio of XM, the deflecting couple acting on the doublet, to the restoring couple brought into play when the axis is deflected through unit angle. As this couple is proportional to the average kinetic energy of the molecules, we have

$$I=NMF(XM/w).$$

Thus we see that for solid and liquids, as well as for gases, I is a function of MX/w.

We need not here go into the question whether the form of the function depends on whether the body is in the solid, liquid or gaseous state. It is sufficient to notice that whatever the state, when $x=0$, $F(x)=0$, and when $x=\infty$, $F(x)=1$.

Thus $F(x)$ will be represented by a curve of the type shown

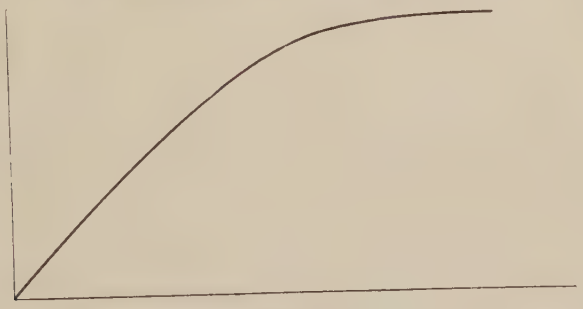


Fig. 1.

in Fig. 1. The force X which occurs in the expression for x is not merely the external electric force acting on the system, the polarised doublets will themselves give rise to strong electric forces and X is the resultant of such forces and the external electric force. We shall take the force due to the polarisation of the doublets as proportional to I and put it equal to kI. Thus if X_0 is the external electric force,

$$X=X_0+kI,$$

and

$$x=\frac{M(X_0+kI)}{w},$$

or

$$I=\frac{w}{Mk}x-\frac{X_0}{k}.$$

This relation between I and x is represented graphically by a

straight line and the value of I corresponding to any value of X_0 can be determined by finding where this line intersects the curve

$$I = NMF(x).$$

The effects corresponding to any finite value of I will be the same as if I doublets per unit volume pointed in the direction of the electric force, while the axes of the rest were uniformly distributed in all directions: and we may picture the substance as containing a number of chains of polarised atoms whose doublets all point in the direction of the electric force as in Fig. 2.

So far as we have gone there has been nothing to differentiate between insulators and metals: in each of these the doublets set under the electric field and give to the substance specific inductive capacity, the value of which is proportional to the value of I when X_0 is unity.

It will be noticed that the electrons in the atoms of the

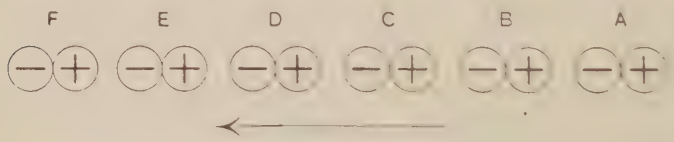


Fig. 2.

substance will be under the influence of forces excited by neighbouring polarised atoms. Thus in the case represented in the figure these forces tend to make the electrons in A move towards B and those in B to C, and so on. On this theory the peculiarity of metals is that electrons, not necessarily nor probably those in doublets, are very easily abstracted by these forces from the atoms when these are crowded together. Thus we may suppose that under these forces an electron is torn from A and goes to B, another from B going to C, and so on along the line, the electrons passing along the chain of atoms like a company in single file passing over a series of stepping stones. Let us suppose that p electrons pass along each of these chains per second, then if there are n of these chains passing through unit area at right angles to the electric force, the current i through unit area will be epn, e being the charge on an electron. If d is the distance between adjacent atoms in the chain, there will be $1/d$ atoms per unit length of chain, and I the number of doublets per unit volume pointing in the

direction of the electric force will be equal to n/d, thus $n=\mathrm{I}d$, and therefore

$$i=ep\mathrm{I}d.$$

The specific conductivity of the metal c is equal to i/X_0, so that

$$c=epd\mathrm{I}/\mathrm{X}_0.$$

The force exerted by the polarised atoms on the nearest electron in a neighbouring atom will be very large compared with that exerted by the external electric force, so that p will be determined by these inter-atomic forces and will not to an appreciable extent depend on the external electric force. The ratio of the current to this force will, therefore, follow the same laws as the ratio of I to the force.

We have seen that the value of I is determined by the intersection of the line,

$$\mathrm{I}=\frac{wx}{\mathrm{M}k}-\frac{\mathrm{X}_0}{k}, \quad \ldots\ldots (1)$$

with the curve

$$\mathrm{I}=\mathrm{NMF}(x), \quad \ldots\ldots (2)$$

where w is the kinetic energy of a molecule, unless the temperature is very low $w=\mathrm{R}\theta$, where θ is the absolute temperature and R the gas constant. When the temperature falls to the stage where the specific heat diminishes with the temperature w will be smaller than the value given by this equation.

When w/MK is considerable the line (1) will be steep and will intersect the curve near the origin, where it approximates to the straight line,

$$\mathrm{I}=\mathrm{NM}x\mathrm{F}'(0). \quad \ldots\ldots (3)$$

The intersection of (1) and (3) is given by

$$\mathrm{I}=\frac{\mathrm{NM}^2\mathrm{F}'(0)\mathrm{X}_0}{w-k\mathrm{NM}^2\mathrm{F}'(0)},$$

and i the current by

$$i=\frac{epd\mathrm{NM}^2\mathrm{F}'(0)\mathrm{X}_0}{w-k\mathrm{NM}^2\mathrm{F}'(0)}.$$

Thus the current is proportional to X_0 and Ohm's law holds. The specific resistance σ is given by the equation

$$\sigma=\frac{w-k\mathrm{NM}^2\mathrm{F}'(0)}{epd\mathrm{NM}^2\mathrm{F}'(0)}.$$

Now, except at very low temperatures, w is equal to $R\theta$, so that σ is expressed by an equation of the form,

$$\sigma = A/\theta - b,$$

It is thus a linear function of the temperature, which is very approximately true for pure metals.

Super Conductivity.

Let us now consider what happens when the temperature is diminished. The slope of the line (1) continually decreases and the intersection of this line with the curve gets farther and farther away from the origin. When the intersection comes on a part of the curve at an appreciable distance from the tangent

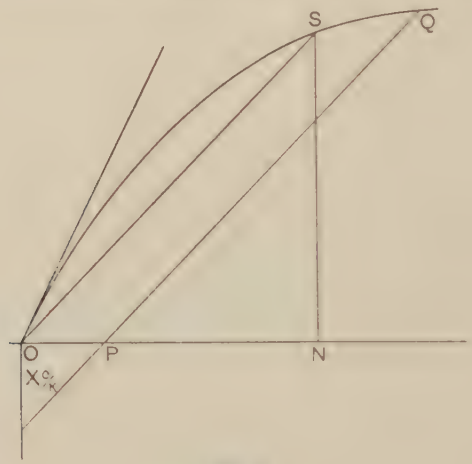


Fig. 3.

at the origin, Ohm's law will no longer hold. Suppose that the slope of the line (1) has fallen so that, as in Fig. 3, it is less than that of the tangent at the origin to the curve $I = NMF(x)$; and after the application of a force X_0, suppose the force is gradually removed, the value of I corresponding to the diminished force will be got by drawing parallels to PQ, continually getting nearer to the origin, and its value when the force has been entirely removed, by drawing a parallel through the origin itself. We see from the figure that in this case the line through

the origin will intersect the curve again at S, showing that I retains the finite value SN after the electric force has disappeared. From the point of view of this Paper, however, the part played by the electric force in metallic conduction is to polarise the metal, *i.e.*, to form chains. When once these are formed, the electricity is transmitted along them by the forces exerted by the atoms on the electrons in their neighbours. Thus, if the polarisation remains, after the electric force is removed, the current will remain, too, just as it did in Kamerlingh Onnes' experiment with the lead ring. We see that we shall have the current remaining after the removal of the electric force, *i.e.*, the metal will be in the super-conducting state as soon as the slope of the line is less than that of the tangent at the origin to the curve, *i.e.*, when

$$\frac{w}{k\mathrm{M}} \text{ is less than } \mathrm{NMF}'(0),$$

or

$$w \text{ less than } \mathrm{NM}^2k\mathrm{F}'(0).$$

Thus, the temperature at which the metal passes into the super-conducting state is such that

$$w=\mathrm{NM}^2k\mathrm{F}'(0). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

kNM is the electric force exerted by the doublets when they all point in one direction: if we denote this force by P, then w_0, the value of w at the critical temperature, is given by

$$w_0=\mathrm{MPF}'(0).$$

If the specific heat of the metal had not commenced to diminish at this temperature, θ_0, the temperature of transition into this state would be given by the equation,

$$\mathrm{R}\theta_0=\mathrm{NM}^2k\mathrm{F}'(0)=\mathrm{MPF}'(0).$$

As, however, the transition takes place at very low temperatures, when the specific heats are variable and w no longer equal to $\mathrm{R}\theta$, we must use a more general expression for w in terms of θ to determine the critical temperature. The persistence of the chains after the removal of the electric force is due to the disturbance due to thermal agitation being too weak to break up the chains when once they are formed. The chains are held together by the electric force due to the doublets in the chain itself as well as by the external electric force, and when we approach the critical temperature the force due to

the doublets is much greater than that due to the external field. We see this from the expression,

$$I=\frac{NM^2F'(0)}{w-NM^2kF'(0)}X_0.$$

This may be written as

$$I=\frac{1}{k}\frac{w_0}{w-w_0}X_0,$$

or

$$\frac{kI}{X_0}=\frac{w_0}{w-w_0}.$$

Now, kI is the part of the force on a doublet due to the other doublets, and we see from this expression that when w is nearly equal to w_0, kI is very large compared to X_0, so that the removal of X_0 will not appreciably weaken the coherence of the chains. On the other hand, at temperatures considerably above the critical, kI is small compared with X_0, so that the external force is essential for the coherence of the chains.

If the disturbing effect on the chains is entirely due to the thermal energy, and if this energy vanishes at the zero of temperature, it will always be possible to find a value of w which satisfies equation (4) and there will always be a critical temperature; *i.e.*, the metal will be able to pass into the superconducting state. It is possible, however, that the action of adjacent atoms may, independently of thermal agitation, tend to make the axes of the doublet take up a definite distribution of orientation giving a kind of crystalline arrangement to the metal, and that the doublets when disturbed from this alignment come under the action of couples tending to restore them to their original positions. We can easily take this into account. All that we have to do is to replace w in the preceding equation by $w+D$, where D is proportional to the restoring couple for unit angular displacement, due to the mutually directive action of the atoms.

The equation to the straight line (1) is now

$$I=\frac{w+D}{Mk}x-\frac{X_0}{k}. \quad \ldots\ldots\ldots \quad (5)$$

We should expect the directive force either to be independent of the temperature or to vary but slowly with it. In this case the slope of the line will not diminish indefinitely as the temperature, but will reach a minimum value whose tangent is D/Mk. If this slope is greater than that of the tangent to the

curve at the origin, whose tangent is $NMF'(0)$, there will be no critical temperature; hence the condition for a critical temperature is

$$D \text{ less than } NM^2kF'(0).$$

The restoring couple may be regarded as due to a local electrical force L exerted by the neighbouring molecules. D will then be LM, and the preceding condition may be expressed as

$$L \text{ less than } PF'(0).$$

w_0' the value of w at the critical temperature is now

$$w_0' = NM^2kF'(0) - D = M(PF'(0) - L).$$

When the slope of the line is considerable, we have from equations (3) and (5)

$$I = \frac{1}{k} \frac{w_0 X_0}{w + D - w_0},$$

or

$$X_0 = \frac{k(w + D - w_0)}{w_0} \frac{i}{epd}, \quad \ldots\ldots (6)$$

and σ, the specific resistance, is equal to

$$\frac{k(w + D - w_0)}{w_0} \frac{1}{epd}. \quad \ldots\ldots (7)$$

Unless the temperature is very low, we may put $w = R\theta$, and we have

$$\sigma = k \frac{(R\theta + D - w_0)}{w_0} \frac{1}{epd},$$

if σ_0 is the resistance at 0°C. and α the temperature coefficient of the resistance,

$$\sigma_1 = \sigma_0(1 + \alpha t),$$

where t is the Centigrade temperature. Comparing this with the previous expression, we see that

$$\alpha = \frac{1}{273 + \frac{D - w_0}{R}}.$$

The condition for the existence of a critical temperature is $D < w_0$, *i.e.*, that the temperature coefficient of the resistance when the temperature is not very low should be greater than 1/273.

When D is considerable the line (5) will be steep, so that at all temperatures the intersection of the curve and the line will

be quite close to the origin. We may, therefore, use equation (3) even for very low temperatures, so that at all temperatures,

$$\sigma = \frac{k}{epd}\,\frac{(w + \mathrm{D} - w_0)}{w_0}.$$

The temperature coefficient of the resistance is proportional to $d\sigma/d\theta$, and this, as we see, is proportional to $dw/d\theta$. This quantity, the rate of increase of the energy with the temperature, is proportional to the specific heat at the temperature θ.

As the specific heats of many substances are very much smaller at the low temperatures obtained by the use of liquid hydrogen or helium, than at normal temperatures, we see that on this theory the temperature coefficients of metals which have no critical temperature ought to be very small at low temperatures. The experiments of Kamerlingh Onnes and Dewar and Fleming show that this is in some cases a very well marked effect. Fig. 4 shows the variation of resistance of

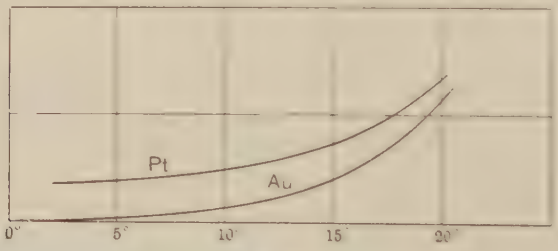


Fig. 4.

gold and platinum suspected of not being quite pure; it will be noticed that at very low temperatures the resistance becomes almost independent of the temperature. Similar effects are shown by many alloys; they would, on the theory, be shown by any metal or mixture which had not too small a value of D and whose specific heat fell appreciably at low temperatures.

In fact, the general behaviour of alloys seems to admit of a satisfactory explanation on the supposition that in them, or at any rate in those whose resistance is considerably greater than the value calculated from their percentage composition, the restoring couple D is much greater than in pure metals. This seems what we might expect when the alloy is not a mere mixture; for if it was a definite compound of the two metals

we should expect that there would be a tendency for the axes of the molecules of one metal to have definite orientation with reference to those of the molecules of the other. The same thing would also apply if the metals did not form definite compounds with each other but did form crystals.

We see from the preceding equations that if D were large for these alloys they would have (1) a small temperature coefficient at normal temperatures and a very small one indeed at temperatures low enough to diminish the specific heats, (2) they would not have a critical temperature and would never pass into the super-conducting state. These are characteristic properties of the resistance of alloys.

Again, if there are m molecules of one metal, n of the other per unit volume we should from the expressions (B) for the specific resistance of a pure metal expect that the specific resistance of the alloy would be given by a formula of the type

$$\sigma=\left\{\frac{mk_1}{ep_1\bar{d}_1}\frac{(w+\mathrm{D}_1-\omega_0)}{w_0}+\frac{nk_2}{ep_2\bar{d}_2}\frac{(w+\mathrm{D}_2-w'_0)}{w'_0}\right\}\Big/(m+n).$$

As this involves the restoring couples D_1, D_2 it cannot be calculated from the resistances of the metals when pure; we see, however, that $\sigma_t-\sigma_\mathrm{T}$ the difference in the specific resistances of the alloy at the temperatures t and T is given by the equation

$$\sigma_t-\sigma_\mathrm{T}=\left\{\frac{mk_1}{ep_1\bar{d}_1}\frac{(w_t-w_\mathrm{T})}{w_0}+\frac{nk_2}{ep_2d_2}\frac{(\omega-\mathrm{T})}{w_0'}\right\}\Big/(m+n),$$

the D's have disappeared from this equation, and it is exactly the value we should have calculated from the resistance of the metals separately. This is the result known as Matthiessen's rule, which states that even when the specific resistance of the alloy can not, the difference between the specific resistance at two temperatures can be calculated from its constituents. So far we have supposed D is independent of the temperature; if it changes appreciably with it, as it might be expected to do if the nature of the compounds, or mixed crystals formed by the two metals, did so, the temperature coefficients would show anomalies such as those found in alloys which have negative temperature coefficients.

I have shown ("Corpuscular Theory of Matter," p. 86) that the electric and thermal conductivities will on this theory bear a nearly constant ratio to each other if the electrons which take part in the conduction are in thermal equilibrium with the metal in their neighbourhood.

ABSTRACT.

The discovery by Kamerlingh Onnes, that at the temperature of liquid helium some metals can exist in a state in which their specific resistance is less than one hundred thousand millionth part of that at 0°C., appears to necessitate the abandonment of the ordinary theory of metallic conduction, as the experimental conditions prohibit the explanation of the phenomenon by an abnormal increase, either in the number or mean free path of the free electrons. The effects observed by Kamerlingh Onnes may, however, be accounted for by a theory of metallic conduction previously given by the author in "The Corpuscular Theory of Matter." On this theory the atoms of some substances contain electrical doublets—*i.e.*, pairs of equal and opposite electrical charges at a small distance apart. The effect of an applied E.M.F. is to alter the heterogeneous distribution of the axes of these doublets by bringing them into partial alignment with the field. The influences preventing complete alignment are considered, and it is shown that if M is the moment of a doublet, N the number per unit volume, w the average kinetic energy of the molecules ($=\Gamma\theta$, except at very low temperatures) and I is the resultant of the molecular moments in the direction of X the electric force on the doublets, then

$$\mathrm{I}=\mathrm{NMF}(\mathrm{XM}/w)=\mathrm{NMF}(x), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in which $\mathrm{F}(x)=0$ when $x=0$, and $\mathrm{F}(x)=\infty$ when x is infinite.

X is made up of the applied electric field X_0 plus an internal field due to the polarised doublets, the latter of which is assumed to be proportional to I. Hence, $\mathrm{X}=\mathrm{X}_0+k\mathrm{I}$, and $x=\mathrm{M}(\mathrm{X}_0+k\mathrm{I})/w$, or

$$\mathrm{I}=wx/\mathrm{M}k-\mathrm{X}_0/k. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

For any value of X_0 the value of I can be found from the intercept of the straight line (2) with the curve (1). The effects due to any value of I will be the same as if I doublets per unit volume pointed in the direction of the field, the axes of the rest being uniformly distributed in all directions, and the substance may be pictured as containing a number of chains of polarised atoms whose doublets all point in the direction of the field. The electrons in the atoms will be acted on by forces due to the neighbouring polarised atoms, and the theory supposes that in conductors the electrons are easily abstracted by these forces from the atoms to which they are attached, and pass, under their influence, from atom to atom round the polarised chain. If p electrons pass along each chain per second, and if there are n chains per square centimetre perpendicular to the field, the current density $i=epn$, where e is the electronic charge. It is shown that p is independent of X_0, and so the ratio of i to X_0 will follow the same laws as that of I to X_0.

When $w/\mathrm{M}k$ is large, as at ordinary temperatures, the slope of (2) will be steep and will intersect (1) near the origin where it approximates to a straight line. In these circumstances it is shown that Ohm's law holds. As the temperature falls the slope of (2) also decreases, and may ultimately become less than that of the tangent at the origin of (1). In this case, if we start with an external field producing a polarisation, I, and gradually reduce the field to

zero, the point of intersection of (1) and (2) moves along the former, but still gives a finite value of I, when (1) passes through the origin—*i.e.*, when $X_0=0$—and a current continues to flow in the absence of an applied E.M.F. as observed in one of Kamerlingh Onnes' experiments.

On this view, therefore, the function of the applied field is to produce the alignment of the doublets; the actual transference of electricity is effected by the large inter-atomic forces brought into being by the polarisation of the doublets. Thus, if the polarisation remains on withdrawing the applied E.M.F. the current will also remain.

In addition to the disturbing effects on the chains due to thermal energy, there may be mutually directive action between different atoms such as gives rise to crystallisation. The effect of this is considered, and it is shown that if this factor is large the metal cannot become superconducting.

DISCUSSION.

Prof. S. P. Thompson, in proposing a vote of thanks to the President for his Paper, said there were one or two points which he had not quite grasped in the course of the lecture, and which he would like to have cleared up. We were asked to assume a ring of polarised atoms with positive and negative sides producing an electric force which caused the transference of electrons from one atom to the next and so on round the circuit, this transference constituting the electric current. Although the responsible factor in the transference of the electrons was the internal force, an external force was necessary to produce this, and he did not see why, on withdrawing the applied field the process should go on instead of stopping after one, or, at most a few, transfers had taken place. Were there any physical grounds for supposing that the forces of restitution were greater in the case of mixed crystals than in homogeneous ones? Lastly, he did not see the physical necessity of introducing the quantity D. Could the total force of restitution from all causes not have been included in a single symbol?

The President, replying to the points raised by Prof. Thompson, said that the energy was not spent in the movements of the electrons, but in creating the initial polarisation, and if, in any circumstances, this could be maintained, the current would go on without any loss of energy, except a little by radiation. The formation of mixed crystals of A and B was due to the force between a molecule of A and one of B being greater than that between two of A or two of B. Hence, the force of restitution called into play on displacing a molecule will be greater in the case of the mixture than in that of the pure metals. He had thought it better to denote the force of restitution due to the directive action of neighbouring molecules, which did not depend much on temperature, by a separate symbol D, and keep it clear from the restoring couple due to the gyrostatic action which did depend on the thermal conditions.

XL. *On an Unbroken Alternating Current for Cable Telegraphy.* By Lieut.-Colonel George O. Squier, *Ph.D.*

Received May 28, 1915.

I. *Introduction.*

The object of this Paper is to propose a new angle of view in the method of transmission of signals in the submarine telegraph cable, and to describe some apparatus for operating on the general principles involved.

Although more particularly an engineering subject, yet it is brought before the Physical Society in the hope that some of its members may see in the plan proposed some points in the fundamental theory of ocean cabling for further research.

The phenomenal progress of wireless telegraphy has been made possible only by the combined efforts of some of the world's best-equipped physicists and the practical engineer. It is in the hope of a similar co-operation of effort for the advancement of cable engineering practice that I invite your attention at this early stage of development.

Experiments have been conducted during the past two years at the works of Messrs. Muirhead & Co. [Ltd.] with a view of determining the practical application of the sine wave type of E.M.F. for cable signalling. The results thus far obtained have only been made possible through the wide range of experience and practice which this distinguished firm of cable engineers have been able to bring to bear on the subject, and they have taken a leading part throughout the experiments carried out.

Sixteen years ago Dr. Crehore and the author conducted some experiments on an Atlantic cable from Waterville, Ireland, to Nova Scotia, Canada, using a special form of dynamo as a source of power for operating the cable.

A transmitter was also devised, in which a special feature was the cutting out or suppressing from the alternating current certain definite semi-waves of current which enabled the cable code to be transmitted.

At that time the demand for increase of speed over ocean cables was not pressing. In fact, the particular cable used was idle for several hours each day, which fact, however, greatly facilitated the opportunities for experimenting.

Wireless telegraphy did not then exist, and the cable relay or amplifier had not appeared, so that very accurate balancing of the duplex-bridge was not required.

Conditions have changed materially since 1899, until at the present moment, it may be said that the matter of obtaining increased speed on ocean cables with present apparatus, is largely controlled by the accuracy with which the duplex-bridge may be balanced.

II. *An Ocean Cable Considered as a Power Line.*

If an engineer were required to design a system for operating an electric motor through an Atlantic cable, no form of generator could be proposed at present, other than a single-phase alternating current of the sine-wave type. This form, we know, will deliver power at the receiving end of the cable more efficiently than any other shape of wave. Furthermore, during the operation of such a motor, the generator would be allowed to run smoothly and regularly, and, in particular, the generator circuit would never be metallically opened or closed during operation.

The opening and closing of an alternating current circuit is well known to produce disturbances of a more or less pronounced character, depending upon the angle of phase at which the current is opened or closed.

Since improvement of the duplex-bridge balance, as stated above, is really at present a most important desideratum, experiments were made using the Muirhead artificial cable to compare the present forms of battery transmitters with various modifications of the sine wave of E.M.F.

In the last analysis it was always the opening and closing of the transmitter circuit which produced the final kick or "jar" in the balance. This is not surprising when we remember that, from an electrical standpoint, few things can be done to a circuit more severe than suddenly to introduce an infinite resistance into it.

However, since there is an appreciable spark or arc at the instant of opening the battery transmitter circuit, this discharge controls to a greater or less degree the character of the break; but the exact influence of this is not so easy to predict, for, though it is probable that the current dies away quicker with a sudden break than it does with a very slow one, which permits the arc to remain for some time, yet it is not proved certainly, for the more rapidly the break is made the faster

the resistance increases, and therefore, probably, the rate of change of current, and with it the counter E.M.F., increases. An increased E.M.F. can bridge a longer gap, but a longer gap may be made in the same time that a short one is, with a less velocity at the break. So it appears that these two considerations counteract each other, and it all depends upon which has the greater influence.

It was only after many variations of the simple alternating current had been tried in the course of experiments to perfect the duplex-balance of the bridge that the fundamental principle of never breaking the transmitter circuit became impressed.

Continuing the analogy of the power plant, it may be remarked that practically every form of cable recorder, amplifier or relay is essentially an alternating-current motor. Its field

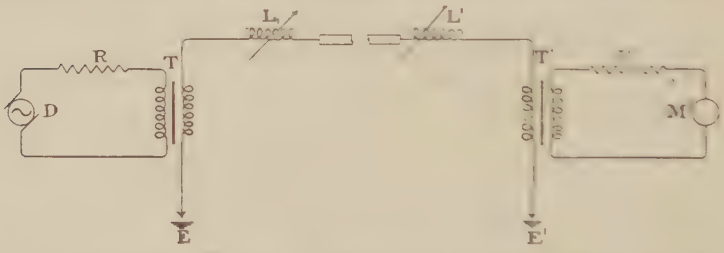


Fig. 1.—Simplex Circuit for Ocean Cable considered as a Power Line.

magnets, armature coil and the counter E.M.F. of damping are subject to the same laws as in the motor for power purposes.

Let us assume, therefore, as a starting point, the standard type of circuit for operating a cable simply as a power plant as the ideal solution of the problem, and then determine by experiment how near this solution may be retained in practice for the purpose of transmitting cable signals according to the present alphabet. In this plan it will be the continued experimental purpose to determine the minimum possible variations in such a system to enable the alternating current received to be interpreted into dots, dashes and spaces.

A siphon recorder placed in the receiving end of such a cable plant would trace on the recording slip an uninterrupted sine-wave of current which may be considered as the theoretically perfect form of siphon record always to be aimed at, although never to be actually attained in practical telegraphy.

In Fig. 1, D represents a single-phase alternating-current

dynamo, R and R′ are resistances and M′ is a motor to be operated.

Since a circuit comprising an ocean cable is largely overloaded with capacity, and already includes a large ohmic resistance, it will be in the direction of increasing the resultant harmonic current flowing in such a circuit, to insert in the sending end of the cable a variable inductance, L_1. For symmetry, L′ is a variable inductance similar to L_1.

T and T′ are iron-cored transformers or auto-transformers, the coils of which are of low resistance, and connected directly to earth at E and E′.

The circuits in Fig. 1 are closed circuits throughout.

For duplex working Fig. 1 becomes Fig. 2, in which L_1, L_2,

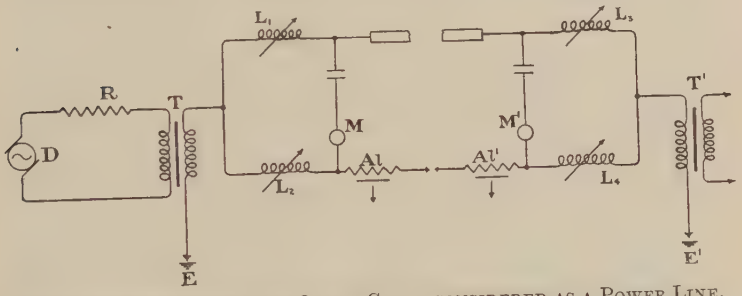


Fig. 2.—Duplex Circuit for Ocean Cable considered as a Power Line.

L_3, L_4 are the inductance arms of the bridge, and the motors M and M′ are inserted in the usual manner for cable working; Al and Al′ are the artificial lines.

It now remains to inquire as to what modifications must be made in the typical duplex power-circuit in Fig. 2 to enable the motor recorders to indicate the elements of the cable alphabet. These elements are three in number, and only three—*i.e.*, the dot, the dash and the space—and each of them is equally important in interpreting the record, and for this reason they will each be considered in the general sense as signal units, rather than the usual way of regarding only the dots and dashes as the signals, and not the spaces.

It is also usual to speak of cable speeds in terms of standard letters per minute transmitted; but for our present purposes it will be more convenient to convert this speed into terms of the frequency of the dynamo. The graph (Fig. 3) exhibits the linear relation between n, the frequency of the dynamo, and letters per minute transmitted, based on the assumption that

the average cable letter, with its space, requires four units of alphabet time.

Careful experiments on an Atlantic cable confirm the theory of the subject—that, no matter what the shape of the alternating current transmitted, approximate sine waves are received at the distant end of the cable. Battery reversals

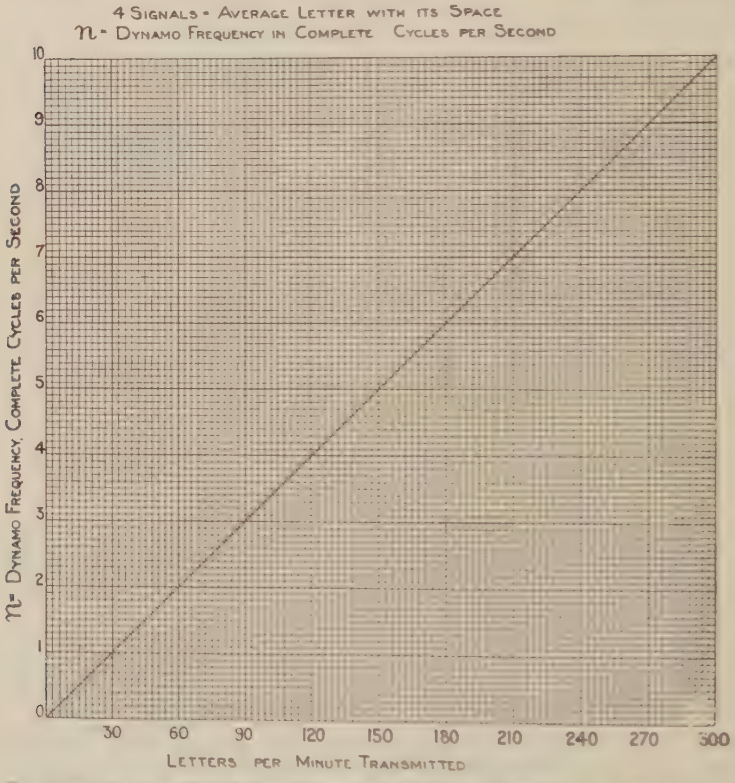


Fig. 3.—Speed of Cable Signalling in Letters per Minute in Terms of the Frequency of the Alternator.

produce just as accurate sine-wave signals on the receiving recorder as does the alternator itself.

In other words, it is the fundamental term of the Fourier analysis which is alone concerned in making the record at the receiving end of the cable.

Since, therefore, both theory and experiment show that a sine form of wave is the only one which can pass through the

cable without changing its characteristic shape, it should be an advantage so to alter the typical power circuit in Fig. 2 as to preserve the sine characteristic as far as possible.

This can be done in a simple manner by operating upon the primary circuit of the transmitter containing the generator, to alter the impedance of the circuit in synchronism with the generator itself. The problem is the more simple from the practical standpoint, because we are dealing with frequencies from about 4 to 10 cycles per second, and, therefore, it is easy to operate with great accuracy on this primary current at any angle of phase. If the change of impedance of the primary circuit always takes place at the instants when the current flowing in the circuit is naturally zero, the fundamental frequency of the current will not be changed, and the sine characteristic of the wave will be very approximately maintained.

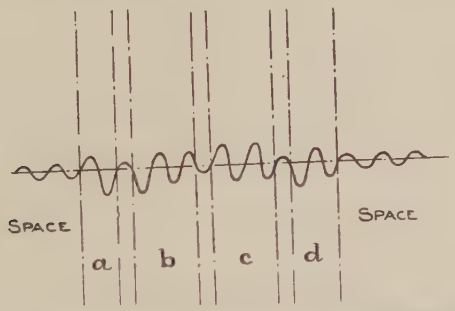


Fig. 4.—Current Waves for Transmitting the Letters *a*, *b*, *c* and *d* by the Alternating Current.

Varying the impedance of the primary circuit will change the amplitude of the individual alternations of the current.

Another way of stating the effect is that the voltage between the end of the cable and the earth is made to vary by this process, to indicate the three elements of the cable alphabet. It should be noted that this plan transmits alternating current for the spaces between letters and words with exactly the same regularity and integrity as for the signals themselves.

Dots, dashes and spaces are each transmitted by impulses of either sign, differing from each other only in amplitude.

Fig. 4 shows diagrammatically the form of current waves transmitted for the letters *a*, *b*, *c* and *d* with the accompanying spaces required.

Although an alternating current operated upon as above

outlined could be read directly from a standard siphon record, yet it would be considered an advantage if the final received record could be printed in Morse characters on the usual receiver slip. To accomplish this it will be necessary to provide an apparatus which operates solely by the amplitude of the current waves received, and not by their sign. Furthermore, since the impulses transmitted and received for the spaces are not required on the printed slip, the apparatus should be capable of sorting out and omitting these parts of the record automatically.

Description of Circuits Used.

Fig. 5 shows diagrammatically one form of circuit arrangement based on the above principles for transmitting and receiving messages. The usual transmitting condensers are shown in the arms of the duplex bridge, instead of the inductances L_1, L_2, L_3, L_4. The ordinary transmitting tape is caused to move synchronously with the generator by being geared directly to its armature shaft. The perforations in the tape are of such a size that the tape advances a distance corresponding to one semi-cycle of the alternator for each individual perforation.

Since in a circuit comprising an ocean cable on which is impressed an alternating E.M.F. the current leads the E.M.F. by an angle which in an infinite cable is constant at 45 deg., it is necessary to provide in the transmitter, mechanism for moving the tape carriage longitudinally relative to the perforations in the tape. A micrometer screw is provided for this purpose, and by its use the current can be operated upon accurately at any angle of its phase, and in practice it is so adjusted that the impedance of the primary circuit will be changed at the instants of zero phase of the current. Since also the angle by which the current leads the E.M.F. is dependent upon the resistance of the primary circuit, there is also provided in addition an adjustment of the transmitter for the very slight difference of phase for the dot-and-dash conditions, so that in effect all operations on the primary current, whether for a dot, dash or space, are adjusted to take place at the zero point of current.

The general operation of the transmitter mechanism is as follows: When no holes are perforated in the tape *t*, the whole resistance R is in the primary circuit and an alternating current is transmitted, which corresponds to spaces between

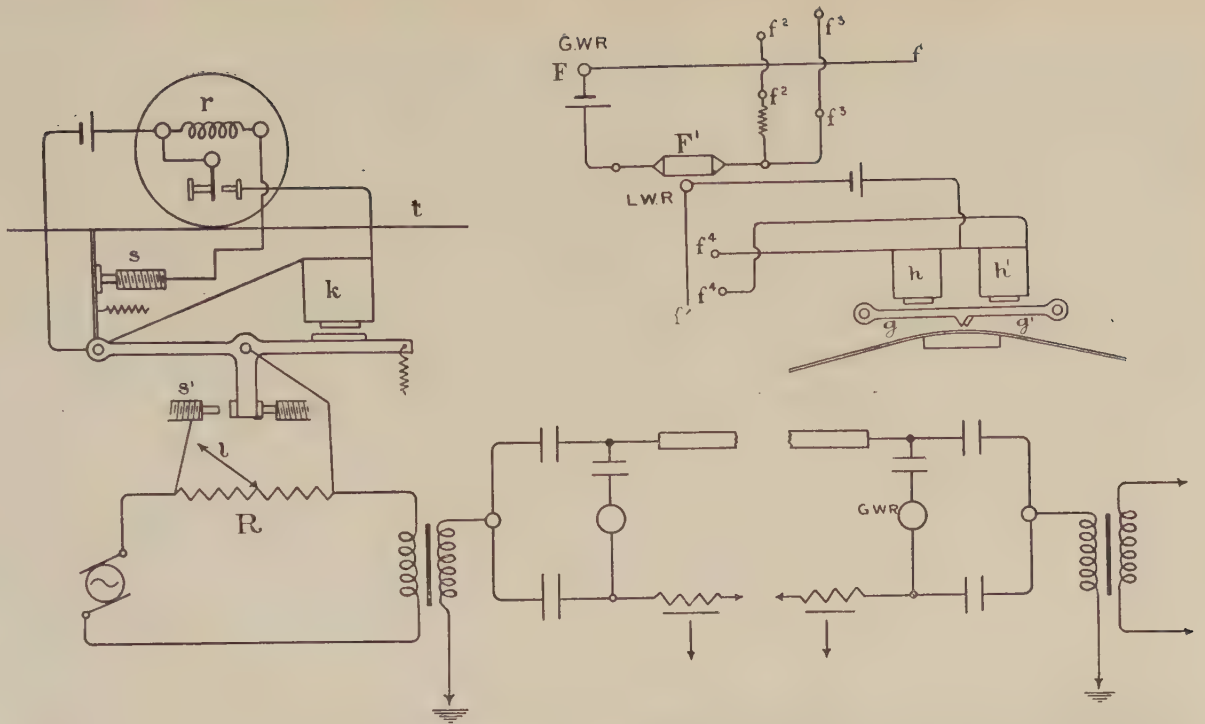


FIG. 5.—ONE FORM OF DUPLEX CIRCUIT ARRANGEMENT FOR TRANSMITTING AND RECEIVING MESSAGES BY THE ALTERNATING CURRENT.

letters and words, and when a perforation in the tape is on the dot side, part of this resistance R is short-circuited, and when a dash perforation occurs, all of this resistance is short-circuited. The relative values of these resistances being adjustable, the amplitude of the individual alternations are under complete control by the ordinary transmitting tape.

In the figure is shown diagrammatically the arrangement for causing a dash impulse to be sent by the transmitter by short-circuiting the whole of the resistance R for a time corresponding to one semi-cycle of the alternator. r is an ordinary relay, k is an electromagnet which operates at the end of each signal or group of signals. The line l leads to an exactly similar arrangement which is provided for the dot impulse by which a certain proportion of the resistance R is short-circuited. In practice these two arrangements are side by side, and in the figure the dot arrangement would be immediately behind the one shown for the dash.

The tape carriage can be moved longitudinally by a micrometer screw which is not shown in the figure, and has an adjustment over a range of a complete semi-cycle. The contacts s and s', and similar ones for the dot mechanism are adjustable by slow-motion screws, so that the instants of contact for a dot and for a dash have an adjustment relatively to each other.

A convenient method of observing the wave-form produced by the transmitter, and for making the above adjustments, is to include an ordinary siphon recorder in the transmitter circuit, or, better, in the cable itself at the transmitting end; by connecting it at the terminals of a very low resistance in the circuit. At the very low frequencies involved, the siphon recorder becomes a most useful and accurate current curve-tracer, and enables the experimenter to observe exactly the shape of wave being transmitted into the cable for any adjustment of the transmitter. Once these adjustments are made for any particular cable they remain unchanged.

Form of Receiving Circuits.

One practical arrangement for receiving the signals on a Morse printer is shown diagrammatically in the upper part of the figure.

F is an adaptation of the well-known Muirhead gold-wire cable relay, in which f is the gold wire which oscillates between the platinum contact posts $f^3 f^3$, about a fixed point at F. A second pair of posts $f^2 f^2$ is provided, and each of these pairs has an

adjustment for altering their distance apart and also for moving them longitudinally along the gold wire and relatively to each other. Instead of being separate contacts, as in the usual case with this relay, each of the pairs are electrically connected as shown. F′ is a local wire relay of similar principle, and f' is its moving part which in its oscillations makes contact with the posts $f^4/^4$ adjustable as above; h and h' are electromagnets for operating the dot and dash arms of the printer; g and g' are two arms pivoted at their outer ends, adapted to make a siphon record at their free ends in the centre upon the moving tape shown beneath. The arm g' is forked to record two ink marks on the tape symmetrically on either side of the central mark made by the arm g, and in line with it across the tape, so that both arms can make a record simultaneously on the tape for a dash.

The incoming alternating current thus causes the gold wire f to oscillate back and forth with different amplitudes depending upon whether dots, dashes or spaces are being received, and the posts $f^2/^2$ are adjusted so that the space amplitude just does *not* make contact, but does make contact for a dot amplitude of either sign. In like manner the posts $f^3/^3$ are adjusted so as just *not* to record a dot impulse, but to make contact for the larger amplitude of a dash of either sign. It is seen that both a dot and a dash contact are made whenever a dash amplitude is received. The flexible gold wire f in its motions about F first strikes the posts $f^2/^2$, and there is a bending or wrapping effect produced in the wire, which for the stronger dash impulses causes contacts with $f^3/^3$ also. The relay F′ operates similarly through a moving arm, making contact with the posts $f^4/^4$ separately or both together, depending upon the amplitude of its swing.

The printer itself becomes an apparatus of marked simplicity, and is nothing more than two small siphons adapted to mark on the slip in the usual manner.

In practice, all adjustments are so made as to provide for transmitting for the spaces as large an amplitude as possible instead of as small an amplitude as possible, in order to approximate more nearly to the ideal electrical conditions for transmission through the cable itself.

In case a cable magnifier such as the Heurtley instrument is used instead of a gold-wire relay, it would be inserted in Fig. 5 in the recorder arm of the bridge, and operate a local wire relay and printer instead of the usual recorder. Since the

relay operates on difference of amplitude of the waves, and not on their absolute value, a cable magnifier would serve to exalt the value of this difference and furnish a greater margin for practical working.

Resolving Power or Definition.

The ideal alphabet to employ in cable signalling would be one in which other things being equal, each letter had the same limit of legibility; because if this is not so, the speed of signalling is lowered to meet the legibility of certain letters only. In present cable practice, what may be called the "resolving power," or the definition, is not equal throughout the letters of the alphabet.

Letters like "a" or "n," for instance, may be considered as perfect letters, and whether sent by one complete cycle of E.M.F. of a dynamo, or by two square-topped waves of equal area and opposite sign separated by a time interval, produce on the siphon record approximate sine-waves. On the other hand, letters like "s" or "h," where three or four square-topped waves of the same sign are sent into the cable consecutively, the received record at high speed fails to resolve these separate impulses, and the siphon record becomes a more or less continuous hump of large amplitude, which the expert operator learns to read without being able to detect the individual impulses; in fact, some practical operators seem to prefer these letters to what are known as "cross-letters," such as "a" and "n."

Careful experiments have been conducted through long cables, which prove, however, that the so called "cross-letters" have a superior legibility, which obviously should be the case from theoretical considerations only.

We may consider the siphon record of an uninterrupted alternating current as possessing 100 per cent definition, independent of the frequency, the voltage employed, or the particular cable used, and regard it as the standard of definition for all signals. Indeed, such a record is, in fact, a message composed of a series of the letter "a" joined together without spaces between.

In present practice it is found necessary to insert a receiving condenser in series with the recorder coil, or an inductive shunt around the coil, for the purpose of improving the definition of certain letters of the alphabet, such as "s" and "h," as well as to eliminate from the record the effects of earth currents of very low frequency induced in the cable.

III. *Electrical Measurements of the Cable Circuits.*

One of the chief attractions in the use of an unbroken alternating current for operating the cable is the fact that for the first time we are enabled to measure the constants of the different elements of the bridge and cable circuits employed under actual signalling conditions. Since one particular frequency is used and the current is never broken, it is necessary and sufficient to use the ordinary commercial hot-wire ammeters and voltmeters which are now available over suitable ranges.

A tachometer is attached to the dynamo armature shaft, so that the frequency is indicated continuously, and these readings, in connection with those of the ammeter and voltmeter, are all that are required to determine the impedance and phase angle of any part of the transmitter circuit, including those of the cable itself.

In the present forms of battery transmitter a square-topped wave is employed, and, in addition, it is found necessary to disconnect the battery and connect the cable to earth during a portion of each individual signal sent. The reason for this is that, in this form of wave, the cable receives a charge dependent upon the time during which the key remains closed for the signal, and when the circuit is opened it is necessary to give time for the cable to become discharged before the succeeding signal can be sent. This discharge is shown in the spark that is seen in the present form of transmitter when the circuit is broken. The percentage of time of each elementary signal during which the cable is connected to earth varies according to the cable, but in long cables it is usually about 25 per cent. of the whole time of the signal.

This means that of necessity the cable is entirely disconnected from the battery at both ends for a period aggregating six hours per day, due to the present method of sending the individual signals.

This discharge of the cable has, indeed, been turned to account for a useful purpose by the late Mr. Gott,* who quite recently developed a system of transmission in which the discharge of the cable is ingeniously utilised to operate the tongue of a relay, which causes the succeeding signal to enter the cable with the opposite sign.

This form of square-topped wave, when combined to make up letters and words, causes a broken and irregular form of

* British patents No. 10,534 and No. 22,364 of 1912.

current to enter the cable, which precludes the possibility of measuring the ordinary quantities, such as voltage, current, impedance and phase-angle of the different elements of the circuit. This is extremely unsatisfactory from an engineering standpoint, and retards progress.

The development of the modern artificial line for simulating the action of the cable has required a large amount of patient and careful work extending over a number of years. As soon as any form of break, with its consequent spark or arc, is removed from the transmitter circuit, and an alternating current of one frequency only is substituted, the artificial line can more faithfully represent the action of the current flowing in the real cable, and its construction may be simplified.

Transmitting Impedance of an Atlantic Cable.

In the case of long submarine cables having resistance and distributed capacity, self-induction and leakance being neglected, we have the well-known sine-wave formulæ for an infinite cable

$$e = \mathrm{E}\varepsilon^{-\sqrt{\frac{\mathrm{CR}\omega}{2}}x} \sin\left\{\omega t - \sqrt{\frac{\mathrm{CR}\omega}{2}}x\right\}, \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$i = \mathrm{E}\sqrt{\frac{\mathrm{C}\omega}{\mathrm{R}}}\,\varepsilon^{-\sqrt{\frac{\mathrm{CR}\omega}{2}}x} \sin\left\{\omega t - \sqrt{\frac{\mathrm{CR}\omega}{2}}x + \frac{\pi}{4}\right\}, \quad . \quad (2)$$

e and i represent instantaneous values of the voltage and current at any point of the cable at a distance x from the origin and a time t: E is the maximum value of the E.M.F. applied to the cable, R and C are the resistance and capacity of the cable per unit length, ω is 2π times the frequency.

These formulæ are accurate for all practical purposes in the case of long cables, as will be shown presently.

At the transmitting end of the cable $x = 0$, and (2) becomes

$$i = \mathrm{E}\sqrt{\frac{\mathrm{C}\omega}{\mathrm{R}}} \sin\{\omega t + 45°\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$i = \mathrm{I}\sin[\omega t + \theta], \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where I is the maximum current and θ is the angle by which the current leads the E.M.F. This angle is constant and equal to 45 deg. The transmitting impedance of the cable is

$$\mathrm{Z}_1 = \frac{\mathrm{E}}{\mathrm{I}} = \sqrt{\frac{\mathrm{R}}{\mathrm{C}\omega}}\sqrt{\frac{\mathrm{R}}{2\pi\mathrm{C}}}\sqrt{\frac{\mathrm{I}}{n}} = c_1\sqrt{\frac{\mathrm{I}}{n}}, \quad . \quad . \quad (5)$$

where c_1 is constant for any particular cable.

$$I=\frac{E}{Z_1}=\frac{E}{c_1}\sqrt{n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Let us assume

$$R=4{,}895 \text{ ohms.}$$

$$C=914\cdot10^{-6} \text{ farads.}$$

$$\text{Length}=2{,}164 \text{ nautical miles,}$$

data which represent one of the best transatlantic cables.

Substituting in (5) and (6) we have

$$Z_1=\frac{923\cdot2}{\sqrt{n}}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

$$I=\frac{E}{923\cdot2}\sqrt{n}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$

The graph (Fig. 6) represents the impedance-frequency curve

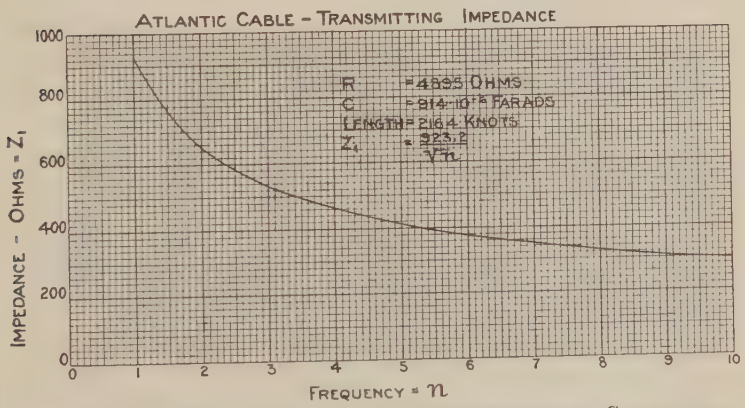


FIG. 6.—TRANSMITTING IMPEDANCE OF A TRANS-ATLANTIC CABLE AS THE SPEED OF SIGNALLING IS VARIED.

at the transmitting end of this particular cable plotted from (7).

For $n=5$, or at 150 letters per minute, the cable impedance is 412 ohms, or only 8·4 per cent. of the ohmic resistance of the cable.

The impedance of this same cable, as measured by the ammeter-voltmeter method, for a frequency of $n=5{\cdot}03$ is 397·5 ohms.

With 50 volts on this cable at the same speed of transmission the current flowing into the cable from (8) is

$$I=0{\cdot}12 \text{ ampere.}$$

The transmitting impedance is most sensitive to change of frequency at the lower frequencies, and by increasing the speed from $n=4$, or 120 letters per minute, to $n=9$, or 270 letters per minute, the cable impedance is decreased in the ratio of 3 to 2, and the current correspondingly increased. The same values of impedance and current deduced above apply to the equivalent artificial line, which is made to balance as far as possible the real cable.

By inserting a hot-wire ammeter in the cable itself beyond the bridge, and also connecting the transmitting end of the cable to earth, through a suitable voltmeter (preferably an electrostatic instrument, to prevent any disturbance of the bridge balance), we have the means at hand for determining the best transmitting conditions for any particular cable and frequency of signalling. Heretofore the value of the transmitting condensers in the duplex arms of the bridge have been more or less arbitrarily assigned by certain practical rules obtained by an expert study of the signals themselves. Here, however, we may approach the problem more scientifically by so adjusting the values of the condensers or inductances, or both, in the bridge branches as to produce a maximum reading of the hot-wire ammeter in the cable itself for any particular voltage assigned. The criterion for best transmitting conditions alone would be that the transmitting impedance should be a minimum, or the current flowing into the cable as measured by a hot-wire ammeter should be a maximum for any particular signalling frequency.

This practical method is, of course, entirely analagous to the present practice of inserting a hot-wire ammeter in the transmitting antenna of a wireless station, except that in the latter case we can go much further and adjust for the maximum current possible at the maximum point of the resonance curve of the antenna. In the cable circuit we are dealing, not with resonance, but with forced waves maintained by the dynamo.

Electrical Stress Upon the Cable.

Submarine cables are at present operated by primary batteries giving an open-circuit voltage of from 50 to 80. The main reason for this is, of course, the constant fear of sub-

jecting the cable itself to undue electrical strain, particularly in the deep sea portion, where repair is difficult and expensive. This is not the only reason, however, although it is the principal one.

The present form of battery transmitter impresses upon the cable square-topped waves, which may be analysed by Fourier's method, and in the hands of Malcolm and a few others the wave-form has been worked out and predicted as the wave passed through the cable to the receiving end. Since we know that it is only the fundamental term of the Fourier analysis which produces an appreciable effect at the receiving end, all of the harmonic waves which are required to build up the square-topped form of wave are impressed upon the cable at the transmitting end, and are absorbed in the cable itself, and never reach the receiving end. These, therefore, represent superfluous electrical charge impressed upon the cable, and this charge for each signal must be got rid of before the succeeding signal can be sent into the cable.

It is, therefore, probable that a practical limit would soon be reached in the present form of transmitter, where the magnitude of this extra charge sent into the cable would become so great that there would be little advantage in further increasing the E.M.F.

The direct influence which increase of voltage produces on the amplitude of the record received as the frequency changes is shown in the graph (Fig. 7). These data were obtained over a submarine cable from New York City to Canso, Nova Scotia, of

$$R = 13{,}700 \text{ ohms.}$$

$$C = 231{\cdot}4 \cdot 10^{-6} \text{ farads.}$$

$$\text{Length} = 880{\cdot}6 \text{ nautical miles.}$$

The cable was used at the transmitting end simplex, without condensers, and at the receiving end the ordinary duplex arrangements were employed with 50 mf. condensers, and the cable recorder was adjusted once for all, and remained unchanged throughout the experiments. The receiving arrangements were not particularly sensitive, and the recorder was not readjusted as the frequency was increased.

It will be observed that, within the limits of these experiments, the amplitude of the excursions of the siphon increased with the voltage for any particular frequency of the dynamo, and as the frequency was increased, the voltage remaining

constant, the double amplitude of the siphon record gradually decreased. Theoretically, these lines are curved lines, but for observations up to 30 volts, with but one adjustment of the recorder, the right lines in this figure best represent the actual observations at each voltage.

It is seen that these lines converge towards a common vanishing point at somewhere about $n = 7$, which means that,

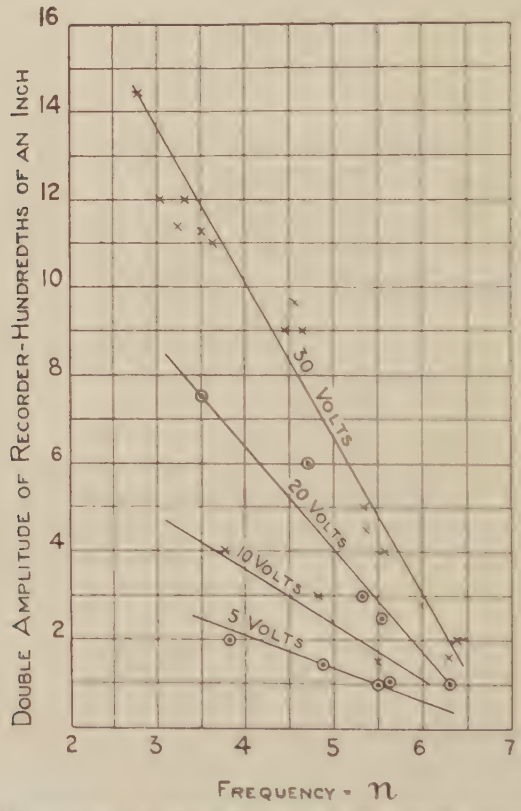


FIG. 7.—VARIATION OF DOUBLE AMPLITUDE OF SIPHON RECORD WITH CHANGE IN FREQUENCY, VOLTAGE CONSTANT.

with the limited sensibility of the particular recorder used (which may be considered as an ammeter if properly calibrated), we cannot expect by any increase of voltage to signal faster than about $n = 7$ with this particular cable.

This graph has been made from data of experiments made

17 years ago, and since that date, due to the great improvement in sensibility of cable relays and amplifiers, the vanishing point at the present day for this cable would be moved to the right along the axis of absissa, to some point corresponding to n equal a considerably larger number than 7.

The above outline will be sufficient to show that in the case of an alternating current being impressed upon a cable, increase of voltage for signalling purposes should be considered from a new point of view, for we know that the power impressed in this case varies directly with the square of the voltage used in transmitting.

Maximum Voltage Along the Cable.

In (1) and (2) above, it is seen from the exponential factor that the maximum ordinates of the E.M.F. and current waves decay according to the logarithmic law, and that the rate of decay is dependent upon the capacity and the resistance of the cable per unit length, and also upon the frequency.

Assuming $n=5$, or 150 letters per minute, the instantaneous value of e or i, and also the maximum volts per volt at sending end have been computed for the Atlantic cable of Fig. 6, and are shown in Table I.

TABLE I.

Frequency=n.	Nautical miles.	e or i.	Maximum volst per volt at sending end.
5	0	1	1
"	135	0·5135	0·5924
"	270	0·1755	0·3510
"	405	0	0·2079
"	540	−0·0616	0·1231
"	675	−0·0630	0·0729
"	810	−0·0432	0·0432
"	945	−0·0222	0·0256
"	1,080	−0·0076	0·0152
"	1,215	0	0·0090
"	1,350	0·0027	0·0053
"	1,485	0·0027	0·0031
"	1,620	0·0019	0·0019

These relations are shown graphically in Fig. 8. It is seen from the table that the maximum voltage impressed upon the cable has decreased to 1/500 of its original value in one complete wave-length, and to 0·5924 of the original value in the first 135 nautical miles from the transmitting end of the cable.

At this speed of signalling a complete wave is 1,620 nautical miles in length; also we have

$$\lambda = 1{,}620 \text{ nautical miles}$$

$$T = \frac{1}{n} = \frac{1}{5} \text{ second.}$$

Total length of cable $= 1{\cdot}33\lambda$.

Rate of propagation of wave through the cable $= \frac{\lambda}{T} = 8{,}100$ nautical miles per second, or the time required for the wave to travel the length of the cable $= 0{\cdot}27$ of a second.

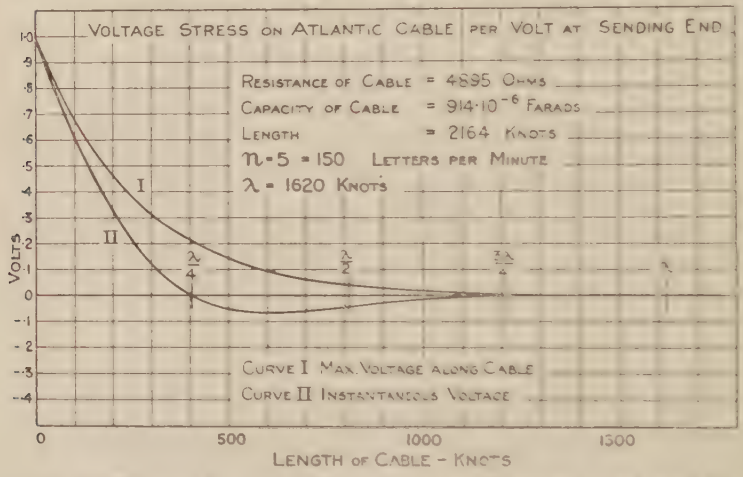


FIG. 8.—VOLTAGE STRESS ON TRANS-ATLANTIC CABLE FOR $n = 5$, OR 150 LETTERS PER MINUTE.

The frequency n has a marked influence upon the rate at which the amplitude of the wave decreases as we proceed along the cable from the transmitting end. The distance at which this amplitude will have $\frac{1}{\varepsilon}$ of its original value is the reciprocal of the coefficient of x in the exponential term of (1), or

$$x' = \sqrt{\frac{2}{CR\omega}} = 258 \text{ nautical miles.}$$

or x' varies inversely with the square root of the frequency. This means that if we increase the speed of signalling from $n = 4$ to $n = 9$ the higher frequency wave will experience the same decrease in voltage in two-thirds of the distance from

the transmitting end. In other words, the higher the speed of signalling attained on any cable for a given transmitting voltage the safer the cable is from electrical strain. Increase in efficiency of the cable plant means increase in safety also.

With a battery, it is possible to subject every centimetre of the cable to the full voltage stress of the battery, as occurs whenever the usual insulation resistance test of the cable is made, or in case the transmitting condenser should be accidentally short-circuited.

With the alternating-current dynamo, however, it is impossible either by accident or by design to produce in the cable in the deep sea portion any voltage, even momentarily, of more than a small fraction of the voltage used at the transmitting end.

This suggests, as has been done before, that, by designing submarine cables with heavier insulation over comparatively short distances at the ends, we can employ higher voltages in signalling with no possibility of subjecting the cable to undue electrical strain at any point of its length.

Submarine cables before laying are at present subjected to a stress of several hundred volts in the cable tanks for a considerable period, and if there is a material advantage in speed in increasing the signalling voltage there would seem to be no good reason for hesitating in the slightest degree to increase the voltage moderately in actual traffic.

Receiving Conditions.

In a long cable there is no reactive influence upon an ammeter or voltmeter placed in any circuit at the transmitting end due to any variation in the form of apparatus inserted at the receiving end of the cable. Opening and closing the cable circuit produces no perceptible indication. The energy of the waves may be considered as leaving the transmitting end of the cable along the conductor path never to return.

We are, therefore, at liberty to consider the electrical conditions for best receiving, separately and independently of the conditions for sending, and then inquire how far these two sets of conditions can be made to harmonize with each other.

Fig. 9 shows one of the standard arrangements for operating on long cables in which the values of the condensers and the resistance of the recorder coil as shown have become more or less fixed quantities in cable practice. The condenser c is limited to this comparatively small value, as its main function

is to improve the definition of letters like "s" and "h," for there is no need to correct for letters like "a" and "n."

This lack of uniform resolving power for all letters, at present limits the free use of any value of condenser in this arm which would be the most suitable from electrical considerations to increase the value of an alternating current in the recorder coil.

In Fig. 9 the attenuated alternating current at the receiving end of the cable is divided at the point B of the duplex bridge, and again subdivided at A and C inversely as the vector-impedances of the paths.

In the recorder arm BC the ideal electrical conditions would be to so assign values to the receiving condenser c, and the inductance and resistance of the recorder coil, as to cause this

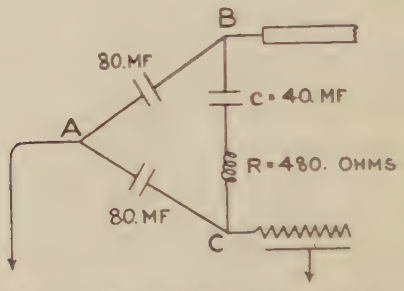


Fig. 9.—Form of Duplex Circuit used on Long Cables.

arm of the bridge to be in resonance, entirely independently of the cable itself, the artificial line, or of any other arm of the bridge. In other words, we should so arrange matters if we could that if the arm BC is short-circuited on itself its natural frequency would be equal to the frequency of the current it is required to receive. In the abstract, there would be an infinite number of solutions for this arm, depending upon the stiffness function $\frac{L}{C}$ selected, each solution fulfilling the condition,

$$CL\omega^2=1.$$

The Cable Dynamo.

The alternator has been designed for the particular purpose of operating a cable, and possesses some novel features. Since we cannot reduce the number of poles to less than two, we must rely on a high magnetomotive force, and a minimum clearance

for the armature to produce, at such low frequencies, the voltage required.

In addition, it is especially desirable to generate an accurate sine wave of E.M.F. These objects have been satisfactorily obtained in the design of the machine used in these experiments.

The resistance of each of the field coils is nearly 1,000 ohms, and the inductance of the armature largely controls the current in the armature circuit, so that the machine can be short-circuited at full voltage without harm.

In starting and stopping the dynamo no form of switch is used, but, instead, the motor which operates the dynamo, starting from rest, gradually builds up the E.M.F. from zero, which gradually dies down to zero at stopping. In this manner there is never any possibility of a "break" in any metallic circuit connected with the transmitting apparatus.

Since the dynamo transmitter comprises the power plant for an ocean cable, it must, therefore, take its place on the instrument table, where the transmitter is directly geared to it. The armature is mounted on ball bearings, and runs with remarkable silence and smoothness under load.

The developments of wireless telegraphy have introduced alternating-current dynamos of considerable power of 100,000 and even 200,000 cycles per second; but for the present purpose we require an alternator to operate at from 4 to 10 cycles per second, and to develop 100 volts or more. This type of dynamo possesses some interest to the engineer, because it may be said to represent the extreme infra-red end of the dynamo spectrum, just as the high-frequency machines mentioned above represent the ultra-violet end. It would be strange indeed if the needs of telegraphy should be ultimately responsible for the development of both ends of this spectrum.

IV. *Summary.*

In the phenomenal development of the wireless art many instruments and methods have been imported from the much older art of wire telegraphy. In like manner may we not, to advantage, re-import some of the methods and instruments developed in wireless telegraphy into the art and science of cable telegraphy?

It has been truly said that "we begin to understand a phenomenon when we can measure it," and for this reason

alone the present plan, it is thought, will appeal to physicists and engineers.

It is seen that once we can employ a system of signalling in which we are dealing with one fundamental frequency only, and the circuit is never broken, the process of measuring the constants of the cable itself and the most suitable bridge arrangements for high efficiency is greatly simplified.

The removal of any form of arc or spark from the cable circuit has been shown to very materially increase the accuracy of balancing the duplex bridge, while the smooth type of waves employed in sending the signals has enabled this important object to be attained with an accuracy greater than has been possible before.

Using the low-resistance coil of a transformer or auto-transformer as the actual source of E.M.F. in the circuit comprising the cable itself, enables each end of the cable to be permanently connected to earth, thus providing a drainage path for earth currents of all kinds induced in the cable, and also adding to its safety.

The present forms of cable recorders, relays and amplifiers, when relieved from the necessity of recording consecutive impulses in the same direction, are liberated from some practical restrictions which have heretofore hampered them.

In the example shown the recorder coil of the gold-wire relay oscillates back and forth at a definite frequency, and the inertia of the moving parts is, therefore, less troublesome, and we are at liberty to increase the flux of the magnetic field, as well as reduce the control of the coil so as to allow it to vibrate more freely at its natural frequency.

No mention has been made of the possible application of this method of transmission to landlines, but it may be remarked that in the present Wheatstone automatic system there is great trouble caused by the induction of one line upon another, and if we could remove from the Wheatstone system the necessity of making and breaking the battery circuit at comparatively high voltage we could, at a single stroke, very largely reduce all such inductive effects.

The more usual method of developing a system of telegraphy is what might be called the "synthetical method"—that is, having certain fundamental principles which are proved to be correct, to endeavour to build up a system in practice which will possess as many as possible of these principles.

This method was employed in the earlier stages of these

experiments, but it was only after abandoning this course altogether, and starting with the ideal solution as outlined above, that it was possible to proceed without radical compromise. It is known, for instance, that certain characteristics are very desirable in any cable system, such as

(*a*) All signals should represent equal lengths of time ;

(*b*) No two consecutive signals should be of the same sign ;

(*c*) The resolving power or definition of each letter of the alphabet should be as nearly as possible the same ;

(*d*) The total quantity of electricity measured in coulombs impressed upon the cable should be as nearly as possible equal to zero for any two consecutive signals.

It will be seen, however, that the ideal solution for power transmission automatically comprises each of these conditions, and, in addition, has other advantages in safety to the cable property from undue electrical strain throughout every centimetre of its length.

ABSTRACT.

1. The Paper proposes a new angle of view in the method of transmission of signals in the submarine telegraph cable, and describes some apparatus for operating on the general principles involved.

2. An ocean cable is considered as a power line, and starting with the standard form of circuit which would be used in case it were required to operate an electric motor through an ocean cable, experiments are described to determine the minimum possible variations required in such a circuit to permit the alternating current received to be interpreted in dots, dashes, and spaces of the present alphabet. The uninterrupted alternating current used in transmission is operated on synchronously by the ordinary transmitting tape, so as to alter the impedance of the transmitting circuit at the instants when the current is naturally zero. Dots, dashes and spaces are each sent by semi-waves of either sign, but of different amplitudes. The alternating current received may be read directly from the record made by a siphon recorder, or this current may be employed to operate a siphon Morse printer, by means of an adaptation of Muirhead's gold-wire relay, or a Heurtley magnifier and a local wire relay.

3. The voltage stress along an Atlantic cable when an alternator is employed is shown, and the transmitting impedance of such a cable is computed as the frequency varies.

4. A special form of cable dynamo to operate at frequencies from 4 to 10 was used in the experiments described.

5. The fundamental principle is developed of never metallically "breaking" the transmitter circuit, which permits of greater accuracy in balancing the duplex bridge.

DISCUSSION.

Mr. W. Duddell said the author and Dr. Crehore some 20 years ago were trying to get rid of the square corners of the waves due to the upper

harmonics. The difficulties now appeared to have been entirely surmounted. As pointed out in the Paper, if the current be broken the harmonics are again introduced, and he thought the method of avoiding this by exact synchronism of the transmitter and generator very ingenious.

Prof. S. P. Thompson thought the great merit of the method of working was that it reduced everything to the sine curve. Anything else than a sine curve was less economical financially and electrically. He admired the author's method of obtaining synchronism, and of making the alterations in amplitude exactly at the zero points.

Mr. A. Campbell said that frequencies of 4 to 10 seemed to be usually employed. What were the limits of frequency practically possible on an Atlantic cable, for example? Was the limit set by the facility with which the signals could be read by the operator?

Dr. H. W. Malcolm thought that by adopting the principle of never breaking the circuit the author was working on the right lines. The distortion in a long cable was so great that the wave-form of the signal received depended very little on the form transmitted, and so the wave-form could be chosen to produce the least disturbance at the transmitting end. For this the use of a series inductance was helpful. Another method was to use a commutator to shut off the recorder while the battery circuit was made or broken.

Prof. G. W. O. Howe asked why the amplitude of the current which passed between signals could not be made zero. Was it an instrumental difficulty, or was there a theoretical advantage in having it large.

The Author, in reply, said it was desirable to have the amplitude large between signals, so as to depart as little as was practicable from its ideal sine wave. Moreover, the energy was utilised to feed the tape. With regard to frequency, 10 was very high from a cable standpoint, but he was hopeful that when the physicist had attended to the problem of the receiving instrument, much higher frequencies, say within the aural limit, would be possible.

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Printed by Geo. Tucker, 1, 2 and 3, Salisbury Court, Fleet Street, London, E.C.

PROCEEDINGS

AT THE

MEETINGS OF THE PHYSICAL SOCIETY

OF LONDON.

SESSION 1914–1915.

October 23, 1914.

Meeting held at the Imperial College of Science.

Prof. Sir J. J. Thomson, O.M., F.R.S., President, in the Chair.

The following Paper was read :—

"Ionisation," Presidential Address by Sir J. J. Thomson.

November 13, 1914.

Meeting held at the Imperial College of Science.

Dr. A. Russell, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. "A Bridge for the Measurement of Self-Induction." By D. Owen, B.A., B.Sc.

2. "On the Coefficient of Diffusion in Dilute Solutions." By B. W. Clack, B.Sc.

November 27, 1914.

Meeting held at the Imperial College of Science.

Dr. A. Russell, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. "Note on the Conduction of Electricity at Point Contacts." By A. F. Hallimond.

2. "The Thermal Conductivity of Badly Conducting Solids." By T. Barratt, B.Sc.

December 18, 1914.

Meeting held at the Imperial College of Science.

Dr. A. Russell, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. "An Exhibition and Description of Some Apparatus for Class Work in Practical Physics." By Dr. G. F. C. Searle, F.R.S.

2.* "A Vacuum Guard Ring and its Application to the Determination of the Thermal Conductivity of Mercury." By H. R. Nettleton, B.Sc.

January 22, 1915.

Meeting held at the Imperial College of Science.

Prof. Sir J. J. Thomson, O.M., F.R.S., President, in the Chair.

The following Papers were read :—

1. "Practical Harmonic Analysis." By Dr. A. Russell, M.A.

2. "Measuring the Focal Length of a Photographic Lens." By T. Smith, B.A.

3. "The Polyscope and its Projection." By Prof. A. W. Bickerton.

* Taken as read.

Annual General Meeting.

February 12, 1915.

Meeting held at the Imperial College of Science.

Dr. A. Russell, M.A., Vice-President, in the Chair.

The Report of the Council was taken as read.

In the year 1914 fourteen ordinary meetings have been held. The meeting on June 20th was noteworthy in that it was held at Cambridge. About 100 members and visitors took advantage of the occasion to visit the works of the Cambridge Scientific Instrument Co., and were subsequently the guests of the Company at a luncheon held by kind permission of the Master and Fellows in the Hall of St. John's College. In the afternoon a science meeting was held in the Cavendish Laboratory, and was followed by tea in the laboratory by the invitation of the President and Lady Thomson. The average attendance at the meetings was 56.

On February 27th the first Guthrie Lecture was delivered by Prof. R. W. Wood, of Johns Hopkins University, Baltimore, his subject being "Radiation of Gas Molecules Excited by Light," and was much appreciated by a large audience.

During the year a report on "Radiation and the Quantum Theory," by Mr. J. H. Jeans, F.R.S., was published by the Society. It is intended that this shall form the first of a series of reports on various branches of physics.

A report of the Committee on Nomenclature and Symbols was published in the August number of the "Proceedings," so that it may be discussed before the Council take any definite action. A second report will be published shortly.

The Annual Exhibition of Apparatus was not held during the past year, as the Council felt it would not be fully supported owing to the war.

The Presidential Address this year was delivered at the first autumn meeting instead of at the Annual General Meeting, as had been the case hitherto, it being felt that the autumn provided a more convenient time of year for a new President to deliver his address.

The number of Ordinary Fellows on the roll at December 31, 1914, as distinct from Honorary Fellows, was 454. Twenty-eight new Fellows have been elected, and there have been eight resignations.

The Society has to mourn the loss of one Honorary Fellow, Prof. W. Hittorf, and six Ordinary Fellows—namely, Mr. R. Kaye Gray, Prof. G. M. Minchin, Prof. J. H. Poynting (one of the Society's Past Presidents), Lieut.-Col. M. T. Sale, Mr. Augustus Stroh, and Sir Joseph W. Swan.

The Report was adopted by the Meeting.

The Report of the Treasurer and the Balance-sheet were presented by the Treasurer.

The total income of the Society again shows an improvement over the preceding year, in spite of the adverse conditions existing in the latter part. It is satisfactory to note that there is a material increase in the income derived from subscriptions and from the sales of publications. The steady increase in the sales of publications seems to indicate that the activity of the Society is being more and more appreciated by the scientific world.

The expenditure for the year has increased, due in part to the greater activity in publication, and also to the more forward policy of the Council with regard to lectures and scientific reports.

The Society's income this year has exceeded its expenditure by £101. 16s. 3d., which I think shows that the Society is in a sound financial condition.

Owing to the favourable conditions which prevailed at the beginning of 1914, the Council was able to order an investment of some of the cash assets, which stood in the last accounts at £698. 14s. I, therefore, purchased on behalf of the Society, £500 India 3½ per cent. Stock.

The total assets of the Society show a slight increase, but too much importance should not be attached to the figures given for the present values of the investments. The values given for the securities were kindly supplied by Messrs. Parr's Bank, and are based on the Stock Exchange minimum prices.

The liabilities on account of the Life Compositions Fund are a little higher, in spite of the deaths of four Life Fellows, as two new Fellows have compounded for their subscriptions. The balance available in the General Fund of the Society has slightly increased since last year.

The Report of the Treasurer was adopted.

The Treasurer in Account with the Physical Society, from January, 1914, to December 31st, 1914.

Dr.	£	*s.*	*d.*	£	*s.*	*d.*
Entrance Fees	26	5	0			
Subscriptions	522	13	6			
Extra Subscriptions	19	19	0			
Arrears	8	8	0			
Subscriptions paid in Advance	6	6	0			
Students' Subscriptions	3	3	0			
Subscriptions for "Science Abstracts" and Advanced Proofs	14	3	6			
				600	18	0
Composition Fees				52	10	0
Dividends :—						
Furness Debenture Stock	15	0	6			
Midland Railway	37	12	6			
Metropolitan Board of Works	6	11	6			
Lancaster Corporation Stock	11	5	0			
New South Wales	8	7	2			
London, Brighton & South Coast Railway	24	2	5			
Great Eastern Railway	18	15	10			
India 3½% Stock	12	6	2			
				134	1	1
Interest on deposit account				6	16	4
Sales of Publications ("The Electrician" Printing & Publishing Co.)				117	10	8
				911	16	1
Transferred from Deposit Account				300	0	0
Balance at Bank, January 1, 1914	98	14	0			
Less Four Cheques	27	7	9			
				71	6	3
				£1,283	2	4

Cr.	£	*s.*	*d.*	£	*s.*	*d.*
"Science Abstracts"				281	11	0
Printing and Distributing Publications of the Society :—						
"The Electrician" Printing & Publishing Co.—						
Proceedings	230	6	4			
Bulletin	24	1	11			
Distribution	47	0	4			
General	80	7	1			
Jeans' Report	40	1	3			
	421	16	11			
Less credit for use of type	2	5	0			
				419	11	11
Periodicals	2	11	4			
Reporting	35	0	0			
Refreshments and Attendance	15	10	0			
Petty cash	23	7	5			
Bank commission	0	6	2			
Royal Asiatic Society	2	2	0			
Guthrie Lecture honorarium	10	0	0			
Jeans' Report honorarium	20	0	0			
				108	16	11
				809	19	10
Invested in £500 India Three and a Half per cent. Stock				445	13	6
Balance at Bank, January 1, 1915	32	8	5			
Less Two Cheques	4	19	5			
				27	9	0
				£1,283	2	4

WILLIAM DUDDELL, *Honorary Treasurer.*

Audited and found correct,
THOMAS H. BLAKESLEY.
J. E. KINGSBURY.

PROPERTY ACCOUNT OF THE PHYSICAL SOCIETY, DECEMBER 31ST, 1914.

ASSETS.

	£	s.	d.
Subscriptions due, Treasurer's estimate	32	0	6
£533 Furness Railway 2 per cent. Debenture Stock	380	0	0
£1,600 Midland Railway 2½ per cent. Preference Stock	944	0	0
£200 Metropolitan Board of Works 3½ per cent. Consolidated Stock	196	0	0
£400 Lancaster Corporation 3 per cent. Redeemable Stock	304	0	0
£254. 2s. 9d. New South Wales 3½ per cent. Debenture Stock	235	0	0
£500 London, Brighton & South Coast Railway Ordinary Stock	490	0	0
£500 Great Eastern Railway 4 per cent. Debenture Inscribed Stock	485	0	0
£500 India 3½ per cent Stock	415	0	0
Balance at Bank	32	8	5
Ditto on deposit	300	0	0
Publications (Treasurer's Estimate)	220	0	0
	£4,033	8	11

LIABILITIES.

	£	s.	d.
Two Cheques	4	19	5
Life Compositions	2,102	0	0
Balance General Fund	1,926	9	6
	£4,033	8	11

WILLIAM DUDDELL, *Honorary Treasurer*

Audited and found correct,

THOMAS H. BLAKESLEY.
J. E. KINGSBURY.

LIFE COMPOSITION FUND.

	£	s.	d.
170 Fellows paid £10 ...	1,700	0	0
3 Fellows paid £15 ...	45	0	0
5 Fellows paid £21 ...	105	0	0
8 Fellows paid £31. 10s.	252	0	0
	£2,102	0	0

NOTE.—Four Fellows who paid £10 deceased during year 1914.

Two Fellows, one at 20 and one at 30 guineas compounded.

Audited and found correct,

WILLIAM DUDDELL, *Honorary Treasurer.*

THOMAS H. BLAKESLEY.
J. E. KINGSBURY.

The Election of Officers and Council then took place, the new Council being constituted as follows :—

President.—Sir J. J. Thomson, O.M., D.Sc., F.R.S.

Vice-Presidents, who have filled the Office of President.—Prof. G. C. Foster, D.Sc., LL.D., F.R.S. ; Prof. R. B. Clifton, M.A., F.R.S. ; Prof. A. W. Reinold, C.B., M.A., F.R.S. ; Prof. Sir Arthur W. Rücker, M.A., D.Sc., F.R.S. ; Sir W. de W. Abney, R.E., K.C.B., D.C.L., F.R.S. ; Prin. Sir Oliver J. Lodge, D.Sc., LL.D., F.R.S. ; Prof. Silvanus P. Thompson, D.Sc., F.R.S. ; R. T. Glazebrook, C.B., D.Sc., F.R.S. ; Prof. J. Perry, D.Sc., F.R.S. ; C. Chree, Sc.D., LL.D., F.R.S. ; Prof. H. L. Callendar, M.A., LL.D., F.R.S. ; Prof. A. Schuster, Ph.D., Sc.D., F.R.S.

Vice-Presidents.—W. R. Cooper, M.A., B.Sc. ; A. Russell, M.A., D.Sc. ; F. E. Smith, R. S. Whipple.

Secretaries.—S. W. J. Smith, M.A., D.Sc., F.R.S. ; W. Eccles, D.Sc.

Foreign Secretary.—R. T. Glazebrook, C.B., D.Sc., F.R.S.

Treasurer.—W. Duddell, F.R.S.

Librarian.—S. W. J. Smith, M.A., D.Sc., F.R.S.

Other Members of Council.—S. D. Chalmers, M.A. ; Prof. G. W. O. Howe, M.Sc. ; Prof. J. W. Nicholson, M.A., D.Sc. ; Major W. A. J. O'Meara, C.M.G. ; C. C. Paterson ; C. E. S. Phillips, F.R.S.E. ; Prof. A. W. Porter, B.Sc., F.R.S. ; Prof. O. W. Richardson, M.A., D.Sc., F.R.S. ; Prof. the Hon. R. J. Strutt, F.R.S. ; W. E. Sumpner, D.Sc.

Prince B. Galitzin was balloted for and elected an Honorary Fellow of the Society.

The following Papers were read :—

1. "A Galvanic Cell which Reverses its Polarity when Illuminated." By A. A. Campbell Swinton.

2.* "On the Criterion of Steel Suitable for Permanent Magnets." By Prof. S. P. Thompson, F.R.S.

3.† "An Investigation of the Photographic Effect of Recoil Atoms." By A. B. Wood, M.Sc., and A. I. Steven, M.A.

* Read in abstract by the Secretary in the absence of the Author.
† Taken as read.

February 26, 1915.

Meeting held at the Imperial College of Science.

Dr. A. Russell, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. "Magnetic 'Character' Figures, Antarctic and International." By Dr. C. Chree, F.R.S.
2. "The Electrification of Surfaces as Affected by Heat." By Dr. P. E. Shaw.
3. "Electromagnetic Inertia and Atomic Weight." By Prof. J. W. Nicholson, M.A., D.Sc.

March 12, 1915.

Meeting held at the Imperial College of Science.

Dr. A. Russell, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. "The Estimation of High Temperatures by the Method of Colour Identity." By Clifford C. Paterson, A.M.Inst.C.E., and B. P. Dudding, A.R.C.Sc.
2. "The Unit of Candle-power in White Light." By Clifford C. Paterson and B. P. Dudding.
3.* "The Relative Losses in Dielectrics in Equivalent Electric Fields, Steady and Alternating (R.M.S.)." By G. L. Addenbrooke, M.I.E.E.

March 26, 1915.

Meeting held at University College, Gower-street, by invitation of

Profs. J. A. Fleming, F.R.S., and A. W. Porter, F.R.S.

Dr. A. Russell, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. "The Change of Thermal Conductivity with Fusion." By Prof. A. W. Porter, D.Sc., F.R.S., and F. Simeon, B.Sc.

* Taken as read on account of lateness of the hour.

2. "An Instrument for the Optical Projection and Delineation of Physical Curves, such as Hysteresis, Resonance and Characteristic Curves." By Prof. J. A. Fleming, D.Sc., F.R.S.

3. "The Stability of Some Liquid Films." By Dr. P. Phillips and Mr. J. Rose Innes, M.A., B.Sc.

Before and after the Meeting a Demonstration of the Green Flash at the Setting of an Artificial Sun was given by Prof. A. W. Porter and E. T. Paris.

April 23, 1915.

Meeting held at the Imperial College of Science.

Dr. A. Russell, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. "On the Theories of Voigt and Everett regarding the Origin of Combination Tones." By Prof. W. B. Morton, M.A., and Miss M. Darragh.

2. "Experiments on Condensation Nuclei produced in Gases by Ultra-violet Light." By Miss M. Saltmarsh.

3.* "On the Self-induction of Solenoids of Appreciable Winding Depth." By S. Butterworth, M.Sc.

May 14, 1915.

Meeting held at the Imperial College of Science.

Dr. A. Russell, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. "Precision Resistance Measurements with Simple Apparatus." By E. H. Rayner.

2. "Some Novel Laboratory Experiments." By F. W. Jordan, B.Sc.

3.* "On Electrically Maintained Vibrations." By S. Butterworth, M.Sc.

* Taken as read in the absence of the Author.

May 28, 1915.

Meeting held at King's College, Strand, by invitation of Profs. O. W. Richardson, M.A., D.Sc., F.R.S., and J. W. Nicholson, M.A., D.Sc.

Dr. A. Russell, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. "On Numerical Relationships between Electronic and Atomic Constants." By Dr. H. S. Allen, M.A.

2. "On a Method of Calculating the Absorption Coefficients of X-Radiation." By H. Moore, A.R.C.Sc.

Prof. O. W. Richardson showed two experiments illustrating Novel Properties of the Electron Currents from Hot Metals; Prof. E. Wilson described and exhibited the Apparatus used in his recent work on High Permeability in Iron; and Mr. T. R. Merton showed an experiment illustrating the Relative Width of Hydrogen and Neon Lines.

June 11, 1915.

Meeting held at the Imperial College of Science.

Dr. A. Russell, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. "The Coefficient of Expansion of Sodium." By E. A Griffiths and E. Griffiths, M.Sc.

2. "Notes on the Calculation of Thin Objectives." By T. Smith, B.A.

3. "On Tracing Rays through an Optical System." By T. Smith, B.A.

4.* "The Accuracy of the Lens and Drop Method of Measuring Refractive Index." By H. R. Nettleton, B.Sc.

* Taken as read in the absence of the Author.

June 25, 1915.

Meeting held at the Imperial College of Science.

Dr. A. Russell, M.A., Vice-President, in the Chair.

The following Papers were read :—

1. "The Conduction of Electricity through Metals." By the President, Sir J. J. Thomson, O.M., F.R.S.
2. "On an Unbroken Alternating Current for Cable Telegraphy." By Lieut.-Col. G. O. Squier, Ph.D.

NOTICES TO FELLOWS.

The question of using letters to denote Fellowship of the Society has been raised from time to time. The practice of Fellows has varied in the past, and it is obviously advisable that there should be uniformity. In the circumstances, the Council has decided to sanction and adopt the letters F.P.S.L. as the official indication of Fellowship of the Society. Fellows are, therefore, asked to use no other letters for this purpose.

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